Mathematical Reviews

Vol. 17, No. 9 OCTOBER, 1956 pp. 931-1036

HISTORY, BIOGRAPHY

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Natucci, Alpinolo. La genesi combinatoria dell'aritmetica. Giorn. Mat. Battaglini (5) **3(83)** (1955), 301–311 (1956).

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Part I was by F. N. David [Biometrika **42** (1955), 1–15; MR **16**, 781].

★**Braithwaite, R. B. Theory of games as a tool for the moral philosopher.** Cambridge, at the University Press, **1955.** 76 pp. $1.25.

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Gürsey, Feza. Obituary: Albert Einstein (1879-1955). Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 101–104 (1 plate) (1956).

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Krafft, M. Carl Friedrich Gauss. Nuovo Cimento (10) **2** (1955), supplemento, 1221–1255.

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Popa, Ilie. Alexandru and Vera Myller. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti **6** (1955), 1–12 (2 plates). (Romanian)
Lists of their mathematical papers are included.

Pajares, E. Poisson. Gac. Mat., Madrid (1) **7** (1955), 105–108 (1 plate). (Spanish)

★ **Ricci-Curbastro, Gregorio. Opere. Vol. I. Note e memorie.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Roma, 1956. xi + 441 pp. 4000 Lire.
This volume contains Ricci-Curbastro's papers published before 1896 with the exception of two which are to be omitted. In addition there are a list of his publications and Levi-Civita's Commemorazione [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (6) **1** (1926), 555–567].

Mitrinović, Dragoslav. Mihailo Petrović (6-V-1868–8-VI-1943). Nauka i Priroda, Beograd **8** (1955), 277–284. (Serbo-Croatian)

Julia, Gaston. Notice nécrologique sur Frédéric Riesz, Correspondant pour la Section de Géométrie. C. R. Acad. Sci. Paris **242** (1956), 2193–2195.

Lebesgue, Henri. L'oeuvre mathématique de Vandermonde. Enseignement Math. (2) **1** (1956), 203–223.
Lecture given in 1937.

Bottema, O. Van der Woude's eightieth year. Nieuw Arch. Wisk. (3) **4 (1956)**, 2–12 (1 plate). (Dutch)

Aitken, A. C. Obituary: Sir Edmund Whittaker, F.R.S. Nature **177** (1956), 730–731.

Ślebodziński, W. L'oeuvre scientifique de Kazimierz Żorawski. Colloq. Math. **4** (1956), 74–88 (1 plate).
A list of Żorawski's published papers is included.

FOUNDATIONS, MATHEMATICAL LOGIC

★ **Huemmer, W. S. L. Kurze Mitteilung über die Grundlagen der Universal-Algebra.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 109–120. Editions du Griffon, Neuchâtel, 1955.
It appears that this paper is intended as an introduction to a comprehensive treatise whose purpose it is "to reduce the whole of mathematical logic and the whole of [axiomatic] mathematics to a single fundamental structure." The author uses a number of symbols which are presumably defined in the above-mentioned treatise but no concrete results are stated and there are no indications of the deductive methods involved in the enterprise.
A. Robinson (Toronto, Ont.).

Gorskiĭ, D. P. Relations, their logical properties and their significance in logic. Moskov. Gos. Univ. Uč. Zap. Trudy Filos. Fak. **169** (1954), 127–160. (Russian)
This paper offers a competent, if rather elementary (and somewhat repetitious), account of basic properties of relations: reflexivity, symmetry, transitivity, etc. It is of historical interest as the first (and to date only) treatment of mathematical logic in a Soviet philosophical journal by a professional philosopher. Hitherto, all work in mathematical logic in the Soviet Union has been done by mathematicians, quite apart from their colleagues in philosophy. Gorskiĭ, author of a rigidly Aristotelian textbook in formal logic (1954), takes issue with M. S. Strogovič for neglecting relations, attempting to interpret all judgments as attributive judgments or instances of class-inclusion, and reducing arguments of the form "$a>b$, $b>c$, therefore $a>c$" to syllogisms. [Cf. Strogovič, Logika, 3rd ed., Gos. Izdat. Polit. Lit., Moscow, 1949.] Gorskiĭ stresses the objective status of relations, their differences from properties, their importance for deductive inference, proof theory, and the axiomatization of various mathematical disciplines. And he sharply rejects the "idealistic" [Kantian] "logic of relations" of Serrus, Bachelard, et al., which views relations as products of a relating mind.
G. L. Kline (New York, N.Y.).

v. Weizsäcker, C. F. Komplementarität und Logik. Naturwissenschaften **42** (1955), 521–529, 545–555.
The author's thesis is that corresponding to Niels Bohr's theory of complementarity in quantum mechanics, there is a special variety of logic in which the truth values are complex numbers. This complementarity logic is different from the quantum logics proposed by G. Birkhoff and J. von Neumann [Ann. of Math. (2) **37** (1936), 823–843] and by P. Destouches-Février [La structure des théories physiques, Presses Universitaires de France, Paris, 1951; MR **13**, 424].
The proposed new logic is simply sketched rather than completely described and defined. The paper contains much philosophical discussion, followed by a list of seven unresolved problems. There is an appendix dealing with the use of linear operations in Hilbert space for quantum mechanics.
O. Frink (University Park, Pa.).

Pil'čak, B. Yu. On the calculus of problems. Ukrain. Mat. Ž. **4** (1952), 174–194. (Russian)
In the first part of this paper the author proves that every system T of the propositional calculus which satisfies the following conditions (i)–(vi) is equivalent to the intuitionistic calculus. (i) If a formula of the form $A \vee \neg A$ is provable in T, then every formula is provable in T. (ii) The deduction theorem for any number of antecedents. (iii) The rule of modus ponens. (iv) $A \wedge B$ is provable if and only if both A and B are provable. (v) $A \vee B$ is provable if and only if at least one of A, B is provable. (vi) $\neg A$ is provable if and only if a formula of the form $A \rightarrow B \wedge \neg B$ is provable. — The paper contains further a detailed proof for the adequacy of Jaśkowski's matrix for the intuitionistic propositional calculus. A shorter proof has been given by G. F. Rose [Trans. Amer. Math. Soc. **75** (1953), 1–19; MR **15**, 1], but the author's method yields a relatively quick decision method.
A. Heyting (Amsterdam).

Rose, Alan. A single axiom for a partial system of the propositional calculus. Z. Math. Logik Grundlagen Math. **1** (1955), 196–197.
The author observes that one of the two axioms of

Rasiowa [Soc. Sci. Lett. Varsovie. C.R. Cl. III. **40** (1947), 22–37; MR **11**, 303] can be dropped if non-equivalence is defined by $(P=Q)=0$, with a new propositional constant 0. *G. Kreisel* (Princeton, N.J.).

Wang, Shih-Chiang. An axiom system for the propositional calculus. J. Chinese Math. Soc. (N.S.) **2** (1953), 267–274. (Chinese. English summary).

The well-known system of Whitehead-Russell (as simplified by Bernays) and Hilbert-Ackermann has four axioms: (1) $\sim(p\vee p)\vee p$, (2) $\sim p\vee(p\vee q)$, (3) $\sim(p\vee q)\vee(q\vee p)$, (4) $\sim(p\vee r)\vee(\sim(q\vee p)\vee(q\vee r))$. Götlind [Norsk Mat. Tidsskr. **29** (1947), 1–4; MR **9**, 1] replaced (3) and (4) by: (3′) $\sim p\vee p$, (4)′ $\sim(p\vee r)\vee(\sim(q\vee p)\vee(r\vee q))$. The author now proves that (3′) is redundant and that (1), (2), (4′) are independent. This yields an interesting system which dispenses with (3) by a slight modification of (4). The author also remarks on the general possibility of eliminating commutative laws and states a solution to problem 64 of Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR **10**, 673]. The known redundancy (Markov) of axiom B6 in Gödel's monograph [The consistency of the continuum hypothesis, Princeton, 1940; MR **2**, 66] on set theory would appear to be of a similar situation. *Hao Wang.*

Sueki, T. Formalization of modal systems. Rep. Univ. Electro-Commun. **7** (1955), 1–16. (Japanese. English summary)

This paper attempts to formalize the syntax and semantics of a modal system which the writer believes is the most normal system of modalities corresponding to our common sense. *From the author's summary.*

Masani, P. The pedagogical importance of Huntington's axioms for the complex number system. Math. Student **23** (1955), 127–143 (1956).

Kondô, Motokiti. Sur la nommabilité d'ensembles. C. R. Acad. Sci. Paris **242** (1956), 1841–1843.

Though the author's starting point is Lebesgue's notion of "namable" sets [J. Math. Pures Appl. (6) **1** (1905), 139–216, p. 205], his development is closely related to systems of predicative analysis such as Grzegorczyk's (G) [Fund. Math. **41** (1945), 311–338; MR **16**, 891]. The definitions of the present note do not make sense unless subscript 1 on p. 1842, line 14 from the bottom, is a misprint for 2. Accepting this, we have: for two subfields K_0 and K of the continuum (C), $K_0\subset K$, (i) a polynomial inequality $F(x_1\cdots x_m t_1\cdots t_n)\geqq 0$ with coefficients in K_0 is called an elementary expression, x being intended to range over the integers, t over K, (ii) sets defined by quantification over x are (E, K_0)-definable if the t are absent (sets of r-tuples of integers), (S, K_0, K)-definable if some t are present (sets of r-tuples whose elements are integers and numbers of K), (iii) sets defined by quantification over t of an (S, K_0, K)-expression are (P, K_0, K)-definable: alternating quantification with respect to x and t in elementary expressions is excluded, presumably because it yields only (P, K_0, K)-definable sets [cf. Kleene, Trans. Amer. Math. Soc. **79** (1955), 312–340, § 3.8; MR **17**, 4]. If R is the set of rationals, (E, R)-sets are those of the finite Kleene-Mostowski hierarchy, (S, R, C)-sets are the elementarily definable sets of (G), and (P, R, C)-sets are projective sets. *G. Kreisel.*

Kondô, Motokiti. Sur les nombres réels et nommables. C. R. Acad. Sci. Paris **242** (1956), 1945–1948.

A pecular definition of the class $\pi(K_0)$ of (E, K_0)-definable real numbers is given by means of "measurable" (E, K_0)-sets of pairs of integers (Theorem 1 on p. 1946 concerning these sets is wrong). The definition is stated to be equivalent to this: $a\in\pi(K_0)$ if and only if $a=\sum a_n p^{-n}$ (a_n and p are natural numbers, $0\leqq a_n<p$ for $n>0$) and the set of pairs $\langle n, a_n\rangle$ is (E, K_0) definable, cf. (G), p. 317 def. B. K_0 is called a relative continuum if $\pi(K_0)=K_0$ because of Theorem 11 (this theorem is trivially wrong as it stands: we replace German capitals by lower case letters in line 2 on p. 1948): If $\pi(K_0)=K_0$, every bounded non-empty (S, K_0, K_0)-set $(\subset K_0)$ has a least upper bound in K_0. But even this is false: let $K_0=\pi(R)$; then (i) the relation $T(n, c)$ is (S, R, K_0)-definable where $T(n, c)$ means that if $c=\sum_1^\infty c_p 2^{-p}$ then $c_p=0$ if either p is not the Gödel number of an n-sentence (prenex closed formula of Peano's arithmetic with $\leqq n$ quantifiers) or p is the number of a true n-sentence, and $c_p=1$ if p is the number of a false n-sentence [see, e.g., p. 128 and 357 of Kreisel, British J. Philos. Sci. **4**, 107–129 (1953), 357 (1954) [MR **15**, 670] or Grzegorczyk, Fund. Math. **41** (1955), 311–338, p. 327 [MR **16**, 891]: for each n there is a (unique) c [of $\pi(R)$] satisfying $T(n, c)$; (ii) the least upper bound of the (S, R, K_0)-set $(En)T(n, c)$ is a number whose binary expansion yields a truth definition for the whole of Peano's arithmetic, and is therefore not in $\pi(R)$; (iii) $\pi[\pi(R)]=\pi(R)$. *G. Kreisel* (Princeton, N.J.).

Kondô, Motokiti. Sur les analyses relatives. C. R. Acad. Sci. Paris **242** (1956), 2084–2087.

The present note concerns relative continua (r.c.). Since on the author's definition of r.c. [see the preceding review], most of his assertions are implausible or false, e.g. Theorems 3, 5, 6, he probably means this: a subfield K of C is a r.c. if every bounded, non-empty (S, K, K)-set has a least upper bound in K. — A function f is called (S, K_0, K)- or (P, K_0, K)-definable if its graph, the set of pairs $\langle u, f(u)\rangle$ is so definable. The theorems concern (i) relations between (S, K_0, K)- functions and those (P, K_0, K)-sets which are definable by a single quantification over K, and (ii) conditions under which the behaviour of an (S, K_0, K)-function on K_0 determines its behaviour on every (S, K_0, K)-set. The general idea of these notes seems interesting, namely to find simple conditions on K_0 which ensure that the real numbers of K_0, (S, K_0, K_0)- and (P, K_0, K_0)-functions and sets satisfy the (main) theorems of ordinary analysis: in the reviewer's opinion this approach provides a more attractive framework for a predicative analysis than the ramified theory of types. Probably the set of real numbers definable in Kleene's hyperarithmetic hierarchy [see the second preceding review above] merits special attention. *G. Kreisel* (Princeton, N.J.).

DeSua, Frank. Consistency and completeness — a résumé. Amer. Math. Monthly **63** (1956), 295–305.

Mannoury, G. Finitistic-formalistic development of projective geometry. Nieuw Arch. Wisk. (3) **3** (1955), 143–147. (Dutch)

This is a sketch of a finitistic formalist development [in the author's sense of formalist, Nieuw Arch. Wisk. (2) **23** (1951), 219–226] of a 3-dimensional projective geometry. It can be described as a formalization of a combinatorial nature. Well formed formulas are sequences

of upper- and lower-case roman letters, and lower case greek letters (points, lines and planes respectively), together with brackets to form from two well formed formulas a third. The 15 "rules of the game" permit forming new (equivalent) well formed formulas from given ones; the first, for example, allows one to replace "AB" by "d" ("d" new to the proof) and is to be interpreted as stating that a line is determined by two distinct points. Many necessary details are omitted and must be supplied by the reader. *P. C. Gilmore* (University Park, Pa.).

★ **Rosenbloom, P. C. Konstruktive Äquivalente für Sätze aus der klassischen Analysis.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 135–137. Editions du Griffon, Neuchâtel, 1955.

Verf. schlägt vor, einen metrischen Raum R konstruktiv kompakt zu nennen, wenn es eine rekursive Funktion $N(n)$ gibt, und dazu ein konstruktives Verfahren, wodurch man aus jeder Menge von $N(n)$ Punkten in R, zwei Punkte wählen kann, deren Abstand weniger als $1/n$ beträgt. Er zeigt an, dass bei dieser Definition der Kompaktheit der Vitalische Satz aus der Funktionentheorie und der Rellichsche Satz konstruktiv beweisbar sind. Die Beweise werden später gemeinsam mit P. Erdös veröffentlicht. *A. Heyting* (Amsterdam).

★ **Teensma, E. The intuitionistic interpretation of analysis.** Proceedings of the Second International Congress of the International Union for the Philosophy of Science, Zurich, 1954, vol. II, pp. 144–150. Editions du Griffon, Neuchâtel, 1955.

Taking the intuitionistic point of view, the author introduces the notion of a "non-incalculable function-symbol". This is a symbol f, connected with a definition D, such that it follows from D that for any natural number n it is impossible that it is impossible that $f(n)$ is a natural number. He uses non-incalculable function-symbols in formulas as if they were ordinary function-symbols; in the reviewer's opinion this needs further justification. The author contends that important parts of analysis are intuitionistically valid, if real numbers are considered as defined by non-incalculable function-symbols and if double negations are inserted at the proper places. A proof is sketched for the existence in this sense of the least upper bound of a bounded class of real numbers. *A. Heyting* (Amsterdam).

★ **Lamouche, André. La théorie harmonique. Le principe de simplicité dans les mathématiques et dans les sciences physiques.** Gauthier-Villars, Paris, 1955. 481 pp. 1800 francs.

"Le processus qu'applique cette théorie, pour résoudre la double énigme du rôle que joue l'instinct de simplicité dans la genèse des sciences, et de la correspondance entre les mathématiques et les phénomènes réels, consiste: (1) à envisager séparément les deux antinomies simple-composé d'une part, simple-complexe de l'autre; (2) à examiner sucessivement, dans les sciences mathématiques, puis dans les sciences physiques, comment on passe de façon progressive de l'une à l'autre de ces deux antinomies, pour reconstituer (suivant des schémas qui s'avèrent rigoureusement parallèles dans ces deux domaines) les systèmes les plus compliqués à partir des éléments les plus simples."

The author philosophizes upon the concepts of simplicity, number, periodicity, continuity, determinism, and relativity, and gives a number of familiar examples of theory-structure. *C. C. Torrance* (Monterey, Calif.).

See also: Menger, p. 949; Badillo Barallat, p. 1013.

ALGEBRA

Clarke, A. B. A theorem on simple cardinal algebras. Michigan Math. J. 3 (1955–56), 113–116.

It is shown that if a simple cardinal algebra $\mathfrak{A}$ satisfies the condition that $a<\infty b$ always implies that $a\leq nb$ for some finite integer n, then $\mathfrak{A}$ is isomorphic to the subalgebra of the algebra of all non-negative real numbers (including ∞). [For terminology see A. Tarski, "Cardinal algebras", Oxford, 1949; MR **10**, 686.] *B. Jónsson.*

Fadini, Angelo. Nozioni di aritmetica appoggiata ad un'algebra. Ricerca, Napoli **6** (1955), no. 3, 17–31.

★ **Shimbel, Alfonso. Structure in communication nets.** Proceedings of the symposium on information networks, New York, April, 1954, pp. 199–203. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1955.

Soit I un ensemble fini d'indices et $R\subset I\times I$ une relation binaire entre éléments de I "pondérée" par une matrice de „distances" $S=\|\alpha_{ij}\|$ telle que $\alpha_{ij}=0$ si $i=j$, $\alpha_{ij}=\infty$ si $(i,j)\notin R$, α_{ij} nombre réel positif si $(i,j)\in R$. Alors, si la multiplication de telles matrices est définie a partir de „sommes" et „produits" de la manière habituelle, mais en entendant par „somme" de x et y le minimum de x et y et par „produit" de x et y leur somme ordinaire, il existe un entier k tel que $S^k=S^{k+1}$ et la longueur β_{ij} du plus court chemin entre les élements i et j est donnée par $S^k=S^{k+1}=\|\beta_{ij}\|$. [Note du rapporteur: Ceci est à rapprocher d'une proprieté analogue des matrices booléennes qui peut être utilisée dans la théorie des reseaux d'interrupteurs. Voir, par ex., Lunc, Izv. Akad. Nauk SSSR. Ser. Mat. **16** (1952), 405–436; MR **14**, 606.] *J. Riguet.*

Combinatorial Analysis

Kitagawa, Tosio; and Mitome, Michiwo. A contribution to a notation system of the confounded factorial experiments. Bull. Math. Statist. **6** (1955), no. 1–2, 1–10.

This paper explains a new notation introduced by the authors to characterize types of confounding and enumerates factorial designs, where this notation may be used. *H. B. Mann* (Madison, Wis.).

Hartley, H. O.; Shrikhande, S. S.; and Taylor, W. B. A note on incomplete block designs with row balance. Ann. Math. Statist. **24** (1953), 123–126.

Youden [Contributions from Boyce Thompson Inst. **9** (1937), 317–326] constructed balanced incomplete block designs (BIBD) which written as row-by-column designs are balanced with respect to both columns (blocks) and rows. He treated the case when the number of treatments, v, equals the number of blocks, b, and arranged the treatments in such a way that every treatment occurred once in each row. The present authors show that row balancing resulting in equal precision of all treatment comparisons is possible if $b=mv$ (m integral). In this case each treatment occurs m times in every row.

They further show that in all BIBD with $r=mk\pm 1$ a type of balance is possible resulting in treatment comparisons of two different precisions. *D. M. Sandelius.*

Mendelsohn, N. S. The asymptotic series for a certain class of permutation problems. Canad. J. Math. **8** (1956), 234–244.

The class of permutation problems is that where the distribution of "hits", elements in specified position or succession, is a modified Poisson distribution of the form

$$p(r)=e^{-a}\frac{a^r}{r!}\left(1+\frac{c_1}{n}+\frac{c_2}{n(n-1)}+\cdots\right);$$

$p(r)$ is the probability of r hits, n the number of elements permuted. The author determines the coefficients $c_1, c_2, \cdots$ in terms of the coefficients of a recurrence for the polynomial associated with the problem by the method of inclusion and exclusion. More precisely, if

$$P_n(x)=\sum\binom{n}{k}M_{(k)}(n)x^k$$

with $M_{(k)}(n)$ the kth factorial moment of the distribution of hits (with n elements), and the recurrence is $P_n(x)=a_1(x)P_{n-1}(x)+a_2(x)P_{n-2}(x)+\cdots+a_p(x)P_{n-p}(x)$ with $a_k(x)$ a polynomial of degree at most k, then the distribution is of modified Poisson form with parameters $a, c_1, c_2, \cdots$ determined by the coefficients of $a_1(x), a_2(x), \cdots, a_p(x)$. Explicit formulas, too long to quote, are given for a, c_1 and c_2. *J. Riordan* (New York, N.Y.).

See also: Tocher, p. 872; Heler, p. 873; Archbold, p. 886; Kempthorne, p. 984.

Linear Algebra, Polynomials Invariants

Ghosh, N. N. A note on determinants with binomial elements. Bull. Calcutta Math. Soc. **47** (1955), 23–25.

An elementary formula for the minors of the sum of two square matrices, one of which is non-singular, is derived and applied to the metric tensor of Einstein's unified field theory. *M. F. Smiley* (Iowa City, Ia.).

Fan, Ky. Some inequalities concerning positive-definite Hermitian matrices. Proc. Cambridge Philos. Soc. **51** (1955), 414–421.

Let $(i_1, i_2, \cdots, i_k)$ denote a principal submatrix of a positive definite Hermitian matrix H with eigenvalues $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n$. Then

$$\lambda_1\lambda_2\cdots\lambda_{p+k}\leq \det(p+1, p+2, \cdots, p+k)\prod_{i=1}^{k}\frac{\det(i, p+1, p+2, \cdots, n)}{\det(p+1, p+2, \cdots, n)}$$

where $1\leq k\leq p<n$ and $0\leq k\leq n-p$ and for $k=0$ it is understood that $\det(p+1, \cdots, p+0)=1$. For $k=p$, $k=n-p$, the bound for $\det H$ is better than the previously known product of complementary principal minors as is proved by a numerical example. Also, if $H=\begin{pmatrix}A & B\\ B^* & C\end{pmatrix}$ then

$$\det H=(\det C)[\min_{R}\det(A+BR+R^*B^*+R^*CR)]$$

with the minimum attained if and only if $R=-C^{-1}B^*$; this is proved by the unimodular congruence $H\to T^*HT$ with $T=\begin{pmatrix}1 & 0\\ R & 1\end{pmatrix}$. There are other related inequalities and formulations of the results in forms invariant under change of base in the underlying metric vector space.

W. Givens (Knoxville, Tenn.).

Schneider, Hans. The elementary divisors, associated with 0, of a singular M-matrix. Proc. Edinburgh Math. Soc. (2) **10** (1956), 108–122.

The elementary divisors associated with the largest non-negative characteristic root ϱ of a non-negative square matrix P are studied. It is shown that this is equivalent to studying the elementary divisors, associated with 0, of a singular M-matrix, where an M-matrix has non-negative diagonal elements, nonpositive off-diagonal elements and all non-zero characteristic roots have positive real parts. That this definition is equivalent to Ostrowski's [see Comment. Math. Helv. **10** (1937) 69–96; Compositio Math. **9** (1951) 209–226; MR **13**, 524] was proved earlier by the author [J. London Math. Soc. **28** (1953), 8–20; MR **14**, 1055]. A square matrix is said to be in standard form if it is transformed by a similarity permutation to block triangular form where the diagonal blocks are irreducible. The main results are formulated in terms of the numbers $R_{ij}(A, P)$ defined as follows for a real matrix A and a diagonally symmetric partition (A_{ij}), $i, j=1, \cdots, k$: set $r_{ij}(A, P)=0$ if $i\neq j$, $A_{ij}=0$ and $r_{ij}(A, P)=1$ if $i=j$, or if $A_{ij}\neq 0$. Next define $R_{ij}(A, P)=\max r_{ih}r_{hl}\cdots r_{nj}$, the max being taken over all sequences $(i, h, \cdots, n, j)$. It is then proved, among other results, that a singular M-matrix in standard form (A_{ij}) has the elementary divisors associated with 0 all linear if and only if $R_{\beta\alpha}=0$ whenever α, β belong to the set of indices of the singular A_{ii}. Take the special case where the M-matrix is Minkowskian, i.e. the condition concerning the real parts of the characteristic roots is weakened by assuming that all row sums are non-negative; in this case the matrix has a dominant main diagonal. It then follows that the elementary divisors associated with 0 are all linear [for a special case see Ledermann, Proc. Cambridge Philos. Soc. **46** (1950), 581–594; MR **12**, 269]. It is further shown that there is only one elementary divisor associated with 0 if and only if $R_{\beta\alpha}=1$, where α, β belong to the set of indices of singular A_{ii} and $\beta>\alpha$.

It is, however, shown by an example that, in general, the set of indices of the singular A_{ii} and the set of indices of the new zero elements in the matrix do not completely determine the elementary divisors associated with 0.

O. Taussky-Todd (Washington, D.C.).

Kalagastov, V. G. On some classes of matrices with non-negative elements. Vologod. Ped. Inst. Uč. Zap. **11** (1953), 129–158. (Russian)

The theorems go back to Perron [Math. Ann. **64** (1907), 1–76], Frobenius [S.-B. Preuss. Akad. Wiss. **1908**, 471–476; **1909**, 514–518; **1912**, 456–477], Romanovsky [Acta Math. **66** (1935), 147–251] and others. *J. L. Brenner.*

Cimmerman, G. K. Expansion of the norm of a matrix in products of norms of its rows. Mikolaiv. Derž. Ped. Inst. Nauk. Zap. **1953**, no. 4, 130–135. (Ukrainian)

Let P_1: (a_{11}, a_{12}), P_2: (a_{21}, a_{22}) be two points in euclidean 2-space, $A=(a_{ij})$. From the relations

$$\det A=2\times\text{area } OP_1P_2=OP_1\cdot OP_2\cdot\sin\theta_{12},$$

it can be shown that

$$|\det A| \leqq (a_{11}^2+a_{12}^2)(a_{21}^2+a_{22}^2)^{\frac{1}{2}}.$$

The generalization given for an $n \times n$ matrix of complex numbers is

$$|\det AA^*| \leqq \prod_i (a_{i1}a_{i1}^*+a_{i2}a_{i2}^*+\cdots)^{\frac{1}{2}},$$

with equality only when the rows of A are orthogonal in pairs. The reviewer calls attention to the corollary: If $b_1, b_2, \cdots$ are non-negative reals, $V=(v_{ij})$ unitary, then

$$b_1b_2\cdots \leqq \prod_i (v_{i1}v_{i1}^*b_1+v_{i2}v_{i2}^*b_2+\cdots).$$

J. L. Brenner (Pullman, Wash.).

Roy, S. N.; and Sarhan, A. E. On inverting a class of patterned matrices. Biometrika **43** (1956), 227–231.

Actually there are several classes considered, all of which are characterized by rather simple patterns among the elements and having explicit inverses that are also of simple patterns, such as continuants whose elements are simply related. They occur in certain statistical analyses. *A. S. Householder* (Oak Ridge, Tenn.).

Wong, Y. K. Some properties of the proper values of a matrix. Proc. Amer. Math. Soc. **6** (1955), 891–899.

The author studies characterizations of matrices for which the largest modulus of the eigenvalues satisfies $|\lambda_1|<1$. If the matrix (a_{ik}) is non-negative, then the condition $\sum_{i=1}^{k} |a_{ik}|<1$ $(k=1, \cdots, n)$ after a suitable permutation of rows and columns is necessary but not sufficient. If the additional condition $\sum_{i=1}^{n} |a_{ik}| \leqq 1$ $(k=1, \cdots, n)$ is imposed, then it is known that the above condition is sufficient, even for real or complex indecomposable matrices. A more general theorem valid for all matrices is: If $s_k=\sum_{i=1}^{n} |a_{ik}| \leqq 1$ and $\sum_{i=k}^{n} |a_{ik}|<1$ for $k=1, \cdots, n$, then $|a_{kk}|+\sum_{j=k+1}^{n} s_j|a_{jk}|<1$ $(k=1, \cdots, n-1; |a_{nn}|<1)$, and there exists an integer $p \leqq n$ such that the maximum sum of the absolute values of elements in the columns of A^p is <1. This implies $|\lambda_1|<1$. Further, ten equivalent conditions are given for $\lambda_1<1$ for a non-negative matrix. The following lemma is used which is related to a result of J. H. Curtiss [J. Math. Phys. **32** (1954), 209–232; MR **15**, 560]: If A and $(I-A)^{-1}$ are non-negative then $a_{kk}<1$ for all k and the diagonal elements of $(I-A)^{-1}$ are $\geqq 1$. The paper uses some axiomatic treatment on norms of matrices, see also A. Ostrowski [Math. Z. **63** (1955), 2–18; MR **17**, 228]. The idea of using $\sum_{i=1}^{k} a_{ik}$ instead of $\sum_{i=1}^{n} a_{ik}$ also turns up in work of Price [Proc. Amer. Math. Soc. **2** (1951), 497–502; MR **12**, 793]. *O. Taussky-Todd* (Washington, D.C.).

Stojaković, Mirko. On an elementary derivation of Cramer's rule. Bull. Soc. Math. Phys. Serbie **7** (1955), 243–244 (1956). (Serbo-Croatian summary)

Stojaković, M. Sur les propriétés d'une classe de matrices. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 33–36.

Let A be an $n \times n$ matrix and let k be fixed. If all principal $k \times k$ minors of A are positive, all non-principal $k \times k$ minors non-positive, but the kth compound of A is not reducible by a permutation, then every $k \times k$ minor of A^{-1} is positive. [Besides the 13 references given by the author, see Duparc and Peremans, Math. Centrum Amsterdam. Rapport **ZW 1954-006** (1954); MR **15**, 847; D. M. Kotelyanskiĭ, Uspehi Mat. Nauk (N.S.) **8** (1953), no. 4(56), 163–167; MR **17**, 191.] *J. L. Brenner.*

Goddard, L. S. Transition matrices occurring in the theory of Markoff processes. Proc. Cambridge Philos. Soc. **51** (1955), 382–384.

A stochastic matrix S (with non-negative elements and row sum one) is obtained from an arbitrary transition matrix (with $a_{ij} \leqq 0$ for $i \neq j$ and column sum zero) by the formula $S=I-\varepsilon A'$, where $0 \leqq \varepsilon \leqq (\max_i a_{ii})^{-1}$. This correspondence is exploited to obtain known properties of transition matrices from those of stochastic matrices. *W. Givens* (Knoxville, Tenn.).

Gluskin, L. M. Automorphisms of multiplicative semigroups of matrix algebras. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 199–206. (Russian)

The main theorem gives a decomposition of an automorphism of a multiplicative system of matrices into four factors. The following lemmas are of interest. 1) Every idempotent matrix of rank r is similar to diag$(1, 1, \cdots, 0, 0, \cdots)$. 2) A primitive idempotent $(efe=f \Rightarrow f=0, e)$ has rank 1. Orthogonal primitive idempotents can be simultaneously reduced to the canonical orthogonal projections (e_{kk}) by the same similarity transformation. *J. L. Brenner* (Pullman, Wash.).

Wiegmann, N. A. On unitary and symmetric matrices with real quaternion elements. Canad. J. Math. **8** (1956), 32–39.

If A is a real symmetric matrix $(A=A')$, there is an orthogonal matrix O $(OO'=I)$ such that OAO' is diagonal. The natural generalization to the complex field is known to involve hermitian symmetry $(A=A^*, UU^*=I)$. The hypothesis $A=A'$ in place of $A=A^*$ introduces complications discussed in I. Schur, Amer. J. Math. **67** (1945), 472–480 [MR **7**, 234] and the hypothesis $OO'=I$ introduces complications discussed in J. Wellstein [J. Reine Angew. Math. **163** (1930), 166–182]. When the complex field is further extended to the field of real quaternions, there are additional complications, because of the relations $(AB)^*=B^*A^*$, but $(AB)' \neq B'A'$. In the new theories which arise in this case, the author considers certain problems of which the following are typical. 1. Let $U=V'$. If $UU^*=I$, when does $VV^*=I$? [If $UU^*=(UU^*)'$ and $(U^*U)=(U^*U)'$.] 2. If D is diagonal, when is $A=UDU'$ symmetric $(A=A')$? [If the polar matrix of A is real.] *J. L. Brenner* (Pullman, Wash.).

★ **Zia-Ud-Din, M. Lectures on symmetric functions and symmetric functional statistics.** Indian Statistical Institute, Calcutta, 1955. 94 pp. (mimeographed)

The major part of this monograph is devoted to a comprehensive and thorough exposition of the theory of symmetric functions. Let $f(x)=(1-a_1x)\cdots(1-a_nx)$ and put $f(x)^{-1}=h_0+h_1x+\cdots$. The functions h_j are termed complete homogeneous or "Aleph" symmetric functions. The relations between these functions, the elementary symmetric functions and the power sums are studied in the first chapter. The formulae of Newton, Waring, Wronski, etc. are derived. A short study of hypersymmetric functions, that is functions in several sets of variables and symmetric in each set, concludes the first chapter. The second chapter deals with determinantal symmetric functions and determinantal relations involving symmetric functions. These are applied to a variety of problems e.g. certain recursive relations between the Bernoulli numbers and formulae for the number of partitions and divisor sums. The third chapter studies symmetric functions in their relation to the symmetric group and its

characters. In the fourth chapter the author sketches the application of symmetric functions to statistics where they appear as test statistics and estimators and the like. This chapter does not go into detail but rather gives a historical account of the application of the theory to statistics. *H. B. Mann* (Lancaster, Pa.).

Zia-ud-din, M. Development of symmetric functions and symmetric functional sta tistics. Proc. Pakistan Statist. Assoc. 3-4 (1954–1955), 3–104.

This is a printed version, almost unchanged, of the mimeographed notes reviewed above.

Schilt, H. Übersicht über die Nullstellen einer Funktion zweiten und dritten Grades. Elem. Math. 11 (1956), 51–56.

Seres, Iván. Über die Irreduzibilität eines Polynoms. Mat. Lapok 3 (1952), 148–150. (Hungarian. Russian and German summaries)

Schur conjectured that if $a_1, a_2, \cdots, a_L$ are distinct integers then (*) $(\prod_{i=1}^{L}(x-a_i))^2+1$ is irreducible. The author proves that if $L=2^k$ and all the $a-s$ are congruent (mod 2), then (*) is irreducible even if we do not assume that the $a-s$ are distinct. The proof uses the theorem of Schonemann-Eisenstein. (In the meantime the author has proved the conjecture of Schur.) *P. Erdős.*

Uchiyama, Saburô. Note on the mean value of $V(f)$. III. Proc. Japan Acad. 32 (1956), 97–98.

Let $q=p^r$, $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$, where $a_j \in \mathrm{GF}(q)$ and $1<n<p$. Let $V(f)$ denote the number of distinct values $f(x)$, $x \in \mathrm{GF}(q)$. The author has proved [same Proc. 31 (1955), 199–201; MR 17, 30] that

$$\sum_{\deg f=n} V(f)=c_nq^n+O(q^{n-1}),\quad c_n=1-\frac{1}{2!}+\cdots+\frac{(-1)^{n-1}}{n!}.$$

He now proves the following theorem. Under the Riemann hypothesis for L-functions we have

$$\sum_{\deg f=n} V^2(f)=c_n^2q^{n+1}+O(q^n).$$

Thus the variance

$$q^{-n+1}\sum_{\deg f=n}\{V(f)-c_nq\}^2=O(q).$$

L. Carlitz (Durham, N.C.).

Gaeta, Federico. Sul risultante tensoriale. Rend. Mat. e Appl. (5) 13 (1955), 472–494.

Let f_i be forms of degrees m_i $(i=1, \cdots, n+1)$ with $n+1$ indeterminates and let $a_{i_1}\cdots i_{m+1}, \cdots, l_{k_1\cdots k_{m+1}}$ be the coefficients of the general terms of $f_1, \cdots, f_{n+1}$, respectively. If

$$a_{i_1}\cdots i_{m_1}\to a_{i_1}\cdots a_{i_{m_1}}, \cdots, l_{k_1\cdots k_{m+1}}\to l_{k_1}\cdots l_{k_{m+1}}$$

are the symbolic decompositions of the coefficients of $f_1, \cdots, f_{n+1}$, respectively, the author gives

$$R=\prod_{\alpha,\cdots,\lambda}\begin{bmatrix}a & l\\ \alpha & \lambda\end{bmatrix}$$

as a symbolic expression of the resultant of the given forms, where the product is extended to all the symbolic determinants. *P. Abellanas* (Madrid).

See also: Potapov, p. 958; Mel'nikov, p. 966.

Lattices

Hiraguchi, Toshio. A note on Mr. Komm's theorems. Sci. Rep. Kanazawa Univ. 2 (1953), no. 1, 1–3.

It is observed that certain theorems of H. Komm [Amer. J. Math. 70 (1948), 507–520; MR 10, 22] about the dimension of subsets of Euclidean n-space can be generalized to other cardinal products. For instance, if X_s is a chain for each $s \in S$ then the dimension of the cardinal product of the X_s is the number of elements in S. Every partially ordered set P with dimension m is isomorphic to a subset of some cardinal product of m chains. *P. M. Whitman* (Silver Spring, Md.).

★ **Monteiro, Antonio Aniceto. Filtros e ideais. I. [Filters and ideals. I.] 2d. ed.** Instituto de Matemática Pura e Aplicada do Conselho Nacional de Pesquisas, Rio de Janeiro, 1955. 57 pp. (mimeographed)

This is the second edition of a booklet giving an introduction to the basic ideas of the theory of partially ordered sets, lattices, and ideals and dual ideals (filters) in lattices. Among the topics covered are σ-lattices, complete lattices, closure operations, Zorn's lemma, prime ideals, ultrafilters, and equivalence of filter bases. *O. Frink* (University Park, Pa.).

Benado, Mihail. Über eine Frage aus der Theorie der Oreschen Normalitätsbeziehungen. Com. Acad. R. P. Romîne 5 (1955), 1241–1243. (Romanian. Russian and German summaries)

Let $xN_\beta y$, where x and y are elements of a lattice and $x \leqq y$, denote that the chain of elements between x and y is β-normal of the first kind in the sense of Ore [Trans. Amer. Math. Soc. 41 (1937), 266–275]. If $xN_\beta y$ and $uN_\beta a$, where $x \leqq u$, then $(a \cup x)N_\beta(a \cup y)$. In general, the Zassenhaus refinement of two β-normal chains need not be β-normal, and β-normality does not imply unitary normality in the sense of Barbilian. *P. M. Whitman.*

Benado, Mihail. Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. II. Théorie des multistructures. Czechoslovak Math. J. 5 (80) (1955), 308–344. (Russian summary)

[For part I, see same J. 4(79) (1954), 105–129; MR 16, 668.] This is a detailed analysis of properties of multistructures, which generalize lattices by permitting a join to be a minimal (not necessarily least) upper bound (which need not exist), and dually. After several examples, an equivalent definition in terms of properties of the operations $\vee$ and $\wedge$ is given. Projectivity of quotients, modularity, semi-modularity, distributivity, and norms retain many of their properties in lattices, but only partial associativity holds. Modularity suffices for the Schreier refinement theorem to hold. Many of the results derived in detail here have already been announced [particularly in Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 5 (1953), 41–48; MR 16, 668].

P. M. Whitman (Silver Spring, Md.).

Balachandran, V. K. A characterization for complete Boolean algebras. J. Osaka Inst. Sci. Tech. 4 (1952), 39–44.

Reprinted from J. Madras Univ. Sect. B. 24 (1954), 273–278 [MR 16, 439].

See also: Lorch, p. 878; Ruprecht, p. 880 Vagner, p. 942.

Rings, Fields, Algebras

Ribenboim, P. Sur une note de Nagata relative à un problème de Krull. Math. Z. **64** (1956), 159–168.

Krull conjectured [Math. Z. **41** (1936), 665–679] that a domain of integrity which is a primary ring and is completely integrally closed, is a valuation ring. Nagata [Nagoya Math. J. **4** (1952), 29–33; MR **13**, 904] published a counterexample to that conjecture. The author points out that Nagata's argument is not conclusive, and describes a modified counterexample, which he obtains as an intersection of valuation subrings of the field $K(x)$ of rational functions over an algebraically closed field K; the valuations of these rings take rational values, and the essential part of the proof (which is too technical to be summarized here) consists in showing that by suitable choice of these valuation rings, their intersection is primary and is not a valuation ring. A slight modification of the construction yields a similar example in which the valuation rings correspond to valuations with integral values. *J. Dieudonné* (Evanston, Ill.).

Reiner, Irving. Unimodular complements. Amer. Math. Monthly **63** (1956), 246–247.

Generalizing a well-known theorem valid for principal-ideal rings, it is proved that if R is a ring in which classical ideal theory holds [see van der Waerden, Moderne Algebra, T. II, 2nd ed., Springer, Berlin, 1940, p. 84; MR **2**, 120] and if $a_1, \cdots, a_n \in R$ generate the unit ideal in R, then there exists a matrix with elements in R, first row $(a_1, \cdots, a_n)$ and with determinant $=1$. *L. Carlitz.*

Northcott, D. G. A general theory of one-dimensional local rings. Proc. Glasgow Math. Assoc. **2** (1956), 159–169.

Ce mémoire rassemble un bon nombre de résultats sur les anneaux locaux de dimension 1, débarassés de toute hypothèse d'égales caractéristiques. On note Q un anneau local de dimension 1, et $\mathfrak{m}$ son idéal maximal. Etude des diviseurs de zéro. Etude des extensions de Q qui sont des modules de type fini sur Q, et en particulier de la suivante: l'anneau total des fractions S de Q est composé direct d'anneaux artiniens primaires; on note $e_1, \cdots, e_n$ les idempotents correspondants, et on considère l'anneau semi-local $Qe_1+\cdots+Qe_n$. Nombreux calculs de longueurs d'idéaux primaires. Si Q est complet et sans diviseurs de zéro, sa clôture intégrale est un Q-module de type fini et un anneau de valuation discrète. Si Q est un anneau d'intégrité, les valuations de son corps des fractions admettant $\mathfrak{m}$ pour centre sur Q sont discrètes, en nombre fini, et en correspondance biunivoque avec les idéaux premiers isolés de (0) dans le complété $\hat{Q}$. Enfin soit Q un anneau local d'intégrité (de dimension 1) et Q' sa clôture intégrale (dans son corps des fractions); pour que Q' soit un Q-module de type fini il faut et il suffit que la topologie de Q soit induite par celle de Q', ou encore que $\hat{Q}$ soit analytiquement non ramifié (c'est à dire que, dans $\hat{Q}$, (0) soit intersection d'idéaux premiers).
P. Samuel (Clermont-Ferrand).

Neculcea, Mihail. Sur l'algorithme de la division. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 3 (1954), no. 4–5, 43–45. (Romanian. Russian and French summaries).

If $\mathfrak{o}$ is a commutative ring with unit element, in which a certain version of the euclidean algorithm holds, then $\mathfrak{o}$ is a principal-ideal ring. The version of the euclidean algorithm used here is as follows: To any element $a \in \mathfrak{o}$ corresponds a non-negative integer $g(a)$, so that to any $a, b \in \mathfrak{o}$, $b \neq 0$, there exist $q, r \in \mathfrak{o}$, such that $a=bq+r$, and, either $r=0$, or $g(r)<g(b)$. In the proof no explicit use is made of the usual assumptions that $\mathfrak{o}$ contains no divisors of zero, or that $g(ab) \geqq g(a)$. *E. Grosswald.*

Sands, Arthur D. Prime ideals in matrix rings. Proc. Glasgow Math. Assoc. **2** (1956), 193–195.

The following results are obtained. The prime ideals of the matrix ring R_n have the form A_n where A is a prime ideal of the ring R. The prime maximal ideals of R_n have the form A_n where A is a prime maximal ideal of R. If A is a maximal ideal of R, A_n is a maximal ideal of R_n if and only if A is prime. *R. E. Johnson* (Northampton, Mass.).

Herstein, I. N. Jordan homomorphisms. Trans. Amer. Math. Soc. **81** (1956), 331–341.

A mapping φ of a ring R into a ring R' is called a Jordan homomorphism if (1) $\varphi(a+b)=\varphi(a)+\varphi(b)$ and (2) $\varphi(ab+ba)=\varphi(a)\varphi(b)+\varphi(b)\varphi(a)$ for all a and b in R. If the characteristic of R' is not two, then (2) is equivalent to the condition (2') $\varphi(a^2)=\varphi(a)^2$. The ring R' is a prime ring if $xR'y=0$ implies $x=0$ or $y=0$. Every ring without divisors of zero, every simple ring (radical or otherwise), and every primitive ring is a prime ring. The author proves that every Jordan homomorphism of an arbitrary ring R onto a prime ring R' of characteristic not two or three is either a homomorphism or an anti-homomorphism. The paper contains references to previous investigations of Jordan homomorphisms, some of which included proofs of special cases of the theorem just stated. The proof of this theorem is based upon a number of ingenious and natural reductions, and is presented in an entirely self-contained and readable form. The paper concludes with an application of the theorem to the study of additive mappings of a ring R onto a ring R' which preserve nth powers, where n is some fixed integer greater than two.
C. W. Curtis (Madison, Wis.).

Behrens, Ernst-August. Zur additiven Idealtheorie in nichtassoziativen Ringen. Math. Z. **64** (1956), 169–182.

The paper develops an additive ideal theory for arbitrary rings, based upon the known results for commutative and non-commutative associative rings. Principal sources of concepts and methods are McCoy [Amer. J. Math. **71** (1949), 823–833; MR **11**, 311], Curtis [ibid. **74** (1952), 687–700; MR **14**, 127], Murdoch [Canad. J. Math. **4** (1952), 43–57; MR **13**, 618] and Krull [Math. Z. **28** (1928), 481–503; Math. Ann. **101** (1929), 729–744]. Let R denote a (not necessarily associative) ring. For any $a \in R$ and any subset $S \subseteq R$, (a) will denote the smallest ideal of R containing a, while $C(S)$ will denote the complement of S in R. If A and B are ideals of R, then AB is the ideal of R generated by all products ab, with $a \in A$ and $b \in B$, while $A:B$ is the set of all $x \in R$ such that $(x)(b) \subseteq A$ for all $b \in B$. A subset M of R is called an M-system if, for every pair of elements m_1 and m_2 in M, there exist elements $m_1' \in (m_1)$ and $m_2' \in (m_2)$ such that $m_1'm_2' \in M$. (The empty set is also called an M-system.) The ideal P is called prime if $(a)(b) \subseteq P$ implies $a \in P$ or $b \in P$, and R is called prime if (0) is a prime ideal. The prime ideal P containing the ideal A is called a minimal prime ideal of A if there exists no prime ideal Q such that $A \subseteq Q \subset P$. If A is an ideal, the radical $r(A)$ of A is the intersection of all minimal prime ideals of A, while the radical of R is $r(0)$.

The ideal Q is called (right) primary if $(a)(b)\subseteq Q$ and $a\notin Q$ implies that $b\in r(Q)$. If $Q_1, Q_2, \cdots, Q_r$ are primary ideals, $Q_1\cap Q_2\cap\cdots\cap Q_r$ is called irredundant if no Q_k contains the intersection of the remaining Q_j for $j\neq k$. Finally, an intersection of a finite number of primary ideals is called normal if it is irredundant and if no component can be replaced by a proper divisor.

The following results are among those proved. 1. An ideal P is prime if and only if, whenever A_1 and A_2 are ideals of R such that $A_1A_2\subseteq P$, then $A_1\subseteq P$ or $A_2\subseteq P$. 2. Generalization of 1. to the case of $n>2$ ideals A_i, with a fixed, but arbitrary, association in the product. 3. Let P be a subset of R. P is a prime ideal if and only if $C(P)$ is an M-system, while, if $P\supset A$, P is a minimal prime ideal of A if and only if $C(P)$ is an M-system maximal in the property of being disjoint from A. 4. Let A and B be ideals of R. Then $r(A\cap B)=r(A)\cap r(B)$ and $r(r(A))=r(A)$. Also, if $A\subseteq B\subseteq r(A)$, then $r(A)=r(B)$. 5. An element $r\in R$ is in $r(A)$ if and only if, every M-system M containing r is such that $M\cap A\neq\emptyset$. 6. $(R/r(0))=(0)$, and an r-semisimple ring is a subdirect sum of prime rings. Moreover, if R is associative and satisfies the descending chain condition for right ideals, $r(0)$ is the Wedderburn radical. 7. If R satisfies the ascending-chain condition for two-sided ideals, the radical of every ideal of R is the intersection of a finite number of prime ideals and some power of $r(A)$ lies in A. 8. $r(0)\subseteq W(R)$ [where $W(R)$ is the author's previous radical, defined for an arbitrary ring in Math. Ann. **127** (1954), 441–452; MR **15**, 928]. 9. The ideal Q is primary if and only if, for any ideals A and B such that $AB\subseteq Q$, $A\not\subseteq Q$ implies $B\subseteq r(Q)$, and if and only if $Q:B=Q$ for any ideal $B\not\subseteq r(Q)$. 10. Let Q and Q' be primary ideals with $r(Q)=r(Q')$. Then $Q\cap Q'$ is primary and $r(Q\cap Q')=r(Q)$ by 4. Also, if $D=Q_1\cap\cdots\cap Q_r$ is an irredundant intersection of primary ideals, and if not all the $r(Q_i)$ are equal, then D is not a primary ideal. 11. Let $A=Q_1\cap\cdots\cap Q_m$ and $A=Q_1'\cap\cdots\cap Q_n'$ be two normal representations of the ideal A as a finite intersection of primary ideals. Then $m=n$ and, subject to a possible renumbering, $r(Q_i)=r(Q_i')$, $i=1, 2, \cdots, m$. Also, if every $r(Q_i)$ is an intersection of a finite number of prime ideals p_{ik}, $k=1, 2, \cdots, s$, then the minimal prime ideals of A are precisely those among the p_{ik} which are minimal in the set of all p_{ik}.

Some counterexamples and further theorems like 11 are included. It should perhaps also be noted that some of these results were used by the author in a subsequent paper, already in print [ibid. **129** (1955), 297–303; MR **16**, 992]. *R. L. San Soucie* (Eugene, Ore.).

Goheen, Harry. The Wedderburn theorem. Canadian J. Math. **7** (1955), 60–62; corrections, 413.

The stated purpose of the author in writing this note was to provide a simpler and more direct proof of the theorem that all finite division rings are commutative. That this proof is incorrect has been pointed out to the author by W. R. Scott of the University of Kansas (see corrections). *C. C. Faith* (East Lansing, Mich.).

Tai, Zitzung. Pseudokonvergenz und die Perfektheit bewerteter Körper. Acta Math. Sinica **5** (1955), 489–495. (Chinese. German summary)

Die Beziehungen zwischen dem Perfektheitsbegriff Krulls und den von Ostrowski entwickelten pseudokonvergenten Folgen sind hier betrachtet. Als ein endgültiges Resultat beweist der Verfasser: Damit ein bewerterer Körper K perfekt ist, ist notwendig und hinreichend, dass jede pseudokonvergente Folge mit primer Breite in K mindestens einen Pseudolimes besitzt. Als eine unmittelbare Folgerung davon erhalten wir die MacLanesche Charakterization für die Maximalität diskret bewerteter Körper. *Author's summary.*

See also: Wiegman, p. 936; Springer, p. 945.

Groups, Generalized Groups

Stolt, Bengt. Über irreduzible Axiomensysteme, die eine endliche abstrakte Gruppe bestimmen. Ark. Mat. **3** (1955), 113–115.

Stolt, Bengt. Zur Axiomatik endlicher Gruppen. Ark. Mat. **3** (1955), 171–180.

Stolt, Bengt. Über gewisse Axiomensysteme, die abstrakte Gruppen bestimmen. Ark. Mat. **3** (1955), 187–191.

Stolt, Bengt. Zur Axiomatik endlicher Gruppen. II. Ark. Mat. **3** (1956), 229–238.

In his dissertation [Univ. of Uppsala, 1953; MR **15**, 99] the author refined the investigations of Baer and Levi and of Lorenzen into those systems of axioms that characterize group composition. In the third of the notes under review, the author obtains four more such systems, thus leaving in doubt only two of the axiom sets under consideration. The other three notes are devoted to determining those axiom systems that characterize group composition under the additional assumption of finiteness, but not otherwise. *R. C. Lyndon.*

Zeeman, E. C. On direct sums of free cycles. J. London Math. Soc. **30** (1955), 195–212.

The first main result may be restated as follows. For A an abelian group, let $A'=\mathrm{Hom}(A, Z)$, Z the integers. Then (1): if A is free, there is a natural isomorphism $A''\simeq A$. This is generalized to (2): if A, B, C are abelian groups, and A and C are free, there is a natural isomorphism $\mathrm{Hom}(\mathrm{Hom}(A, B), C)\simeq A\otimes\mathrm{Hom}(B, C)$, tensor product. A companion result gives (3): for A, B, C abelian, A or C free, A or C finitely generated, there is a natural isomorphism $\mathrm{Hom}(A, B\otimes C)\simeq\mathrm{Hom}(A, B)\otimes C$. Examples establish the necessity of the hypotheses.

The proof of (1) reduces easily to showing that if ϕ in A'' vanishes on every g in A' with finite support, then $\phi=0$. If the rank $\mathfrak{a}$ of A is finite, this is trivial; Specker [Portugal. Math. **9** (1950), 131–140; MR **12**, 587] proved (1) for $\mathfrak{a}$ countable. The present extension to general $\mathfrak{a}$ makes use of the axiom of choice, and of the following axiom of accessibility, due to Tarski and shown consistent by Shepherdson [J. Symb. Logic **17** (1952), 225-237; MR **14**, 834]: if α is a limit ordinal, and $\beta<\alpha$ implies $2^{\aleph_\beta}<\aleph_\alpha$, then ω_α is the limit of a sequence, of type less than ω_α, of ordinals less than ω_α. The theorem follows easily from the following lemmas, where $\Psi(\mathfrak{a})$ expresses that the theorem holds for A of rank $\mathfrak{a}$. (a) $\Psi(\aleph_0)$; (b) if $\mathfrak{a}<\mathfrak{b}$ then $\Psi(\mathfrak{b})$ implies $\Psi(\mathfrak{a})$; (c) if $\omega_\alpha=\lim_{\gamma<\mu}\gamma_\gamma$, μ, $\gamma_\gamma<\omega_\alpha$, and $\mathfrak{a}<\aleph_\alpha$ implies $\Psi(\mathfrak{a})$, then $\Psi(\aleph_\alpha)$; (d) $\Psi(\mathfrak{a})$ implies $\Psi(2^{\mathfrak{a}})$. The burden of the argument lies in the rather intricate proof of (d). Proof of (2) follows closely that of (1), while (3) is more elementary, not requiring transfinite induction.

R. C. Lyndon (Ann Arbor, Mich.).

Maurer, I. Über die Normalreihen der Gruppe verallgemeinerter unendlicher Permutationen. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 499–505. (Romanian. Russian and German summaries)

Let A^* be the strong direct sum of a countable number of copies of a group A, and let C be that group of automorphisms of A^*, the members of which exchange components in A^*. Then C is a realization of the group of permutations of the natural numbers. Let $S_\infty(A)$ be the relative holomorph of C over A^*, let $S(A)$ be the relative holomorph over A^* of the finite permutation, let $P(A)$ be the relative holomorph over A^* of the even permuations and let $E(A)$ be the subgroup of these holomorphs which corresponds to A^*. For a subgroup X of A, let $E(x)$ be that subgroup of $E(A)$ consisting of all elements of the form $(x, x, \cdots, x, \cdots; \varepsilon)$, where x runs over X and where ε is the identity automorphism from C. Suppose that

$$\alpha: A \supset N_1 \supset N_2 \supset \cdots \supset M \supset M_1 \supset \cdots \supset (e)$$

is a principal series of A where $M \neq (e)$ is a subgroup of the center of A. If $m^{(k)}$ is the typical element of M_k, then the author proves that

$$S_\infty(A) \supset S(A) \supset P(A) \supset E(A) \supset E(N_1) \supset E(N_2) \supset \cdots \supset E(M) \supset E(m) \supset \cdots \supset E(m^{(k)}) \supset \cdots \supset E(e)$$

is principal. The same result holds for composition series if M is merely normal abelian in A. If M has an infinite number of followers in the series α, then the author proves that an infinite number of series of the given type can be constructed in $S_\infty(A)$ from α. Otherwise the number is finite. Satz 2 of the German summary is mistranslated from the original. *F. Haimo* (St. Louis, Mo.).

Szász, F. On groups every cyclic subgroup of which is a power of the group. Acta Math. Acad. Sci. Hungar. **6** (1955), 475–477. (Russian summary)

If, for a natural (possibly 0) integer k, the kth power of a group G is defined to be the subgroup of G generated by all the kth powers of the elements of G, then the author shows that the cyclic groups are characterized by the property that every cyclic subgroup is some kth power of the group. *F. Haimo* (St. Louis, Mo.).

Sultanov, R. M. Decomposition of abelian groups without torsion into a direct sum of cyclic subgroups. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **3** (1948), 65–72. (Russian. Azerbaijani summary)

Sultanov, R. M. Decomposition of primary groups into a direct sum of cyclic subgroups. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **4–5** (1952), 168–173. (Russian. Azerbaijani summary)

Prüfer showed that a countable primary group G with no elements of infinite height splits into the direct sum of cyclic groups [I. Kaplansky, Infinite abelian groups, Univ. of Michigan Press, 1954, pp. 23, 74; MR **16**, 444]. Replace G by a countably generated, torsion-free, left K-module where K is a principal-ideal domain and where the elements of K act as isomorphisms into G. The length of an element of K is to be the number of primes into which that element decomposes. For a submodule F of G and for an element $g \in G$, form the length of the generator of that ideal of K which nullifies the coset gF in G/F and take the sup of these lengths as g ranges over G, and call this sup the height of F. Then the author shows that a necessary and sufficient condition that G split into a direct sum of cyclic submodules is that every submodule with a finite number of generators has finite height. By an example involving a submodule of the direct sum of two copies of the group of rationals which have powers of 2 as their denominators viewed as a module over the integers, he shows that one cannot recapture the above theorem of Prüfer by replacing the condition of finite height for finitely generated submodules by the condition of finite height for each of the elements of G.

In the second paper, he extends the theorem of Prüfer by redefining the height of an element and (implicitly) the order of an element in terms of a fixed prime in K. That is, G is α-primary if there exists an integer $n(x)$ for each $x \in G$ such that $\alpha^{n(x)}x=0$, and x has α-height m if there exists $y \in G$ with $\alpha^m y=x$ while $\alpha^{m+1}z=x$ has no solution $z \in G$. Then he shows that an α-primary left K-module with a countable number of generators splits into a direct sum of cyclic submodules if and only if G has no element of infinite α-height. *F. Haimo.*

★ **Calame, André. Les relations caractéristiques des bases du groupe symétrique.** Thèse, Université de Neuchâtel, 1955. 101 pp.

Let G be a finite group generated by two elements S and T, which are connected by certain relations $g(S, T)=1$. A set of characteristic relations which define G is called independent if no one of the relations is a consequence of the rest; it is minimal if no smaller set of relations defines the group. For example, the relations $S^3=T^2=STST^{-1}=1$ form an independent but not minimal set of relations for the symmetric group $\mathbf{S}_3$, whereas $T^2=TSTS^{-2}=1$ is a minimal independent set. It was shown by S. Piccard that for $n=3, 4, 5$ each pair of generators of the symmetric group $\mathbf{S}_n$ satisfies a minimal set of $n-1$ relations. The author proves the same result for $\mathbf{S}_6$, and shows that E. H. Moore's set of defining relations for $\mathbf{S}_n$ [Proc. London Math. Soc. **28** (1897), 357–366] is not minimal. Starting with the cyclic subgroup H generated by some element S the author reconstitutes the group $\mathbf{S}_6$ by forming successively the cosets HT^j, HT^jS^k, $HT^jS^kT^r$, etc. Of the 313 cosets so obtained with an S of order 4, 180 were distinct. The 133 duplicates of previous ones are not used to form further cosets, but serve with the relations $S^4=T^3=1$ to produce 135 characteristic relations, of which every possible relation between S and T is proved to be a consequence. These relations are then reduced to a minimal independent set of five. Under inner automorphisms each pair of generating elements for $\mathbf{S}_6$ is equivalent to one of 163 pairs, which are further grouped into classes by a type of birational functional equivalence. For each class five relations suffice to define the group. *J. S. Frame.*

Huppert, Bertram. Über die Auflösbarkeit faktorisierbarer Gruppen. III. Math. Z. **64** (1956), 138–148.

[For the first two papers of this series (the second jointly with N. Itô) see Math. Z. **59** (1953), 1–7; **61** (1954), 94–99; MR **15**, 197; **16**, 332.] The numbering of theorems is continued. Theorem 4 of the second paper stated: If a finite group G is factorizable $G=LN$, where L has a cyclic normal subgroup of index 2 and N is nilpotent, then G is soluble. If the index 2 is replaced by an arbitrary prime index, the theorem is no longer valid; for the simple group of order 168 can be factorized into the product of a nonabelian group of order 21 (which has a cyclic normal subgroup of index 3) and a dihedral group of order 8. This counterexample is in a way unique. Theorem 6: If a non-soluble group G is factorizable $G=LN$, where L

is of order pq (p and q primes), and N is nilpotent, then G has a subgroup H that can be mapped homomorphically onto the simple group of order 168. If in Theorem 4 a little more is assumed about N, then a little less need be assumed about L. Theorem 7. If $G=LA$, where L has a cyclic normal subgroup of prime index and where A is abelian, then G is soluble. Here it is not sufficient to assume that L is metacyclic — the symmetric group S_5 provides a counterexample; for $S_5=LZ$, where L is the normalizer of a 5-Sylow subgroup (and has a cyclic normal subgroup of order 5 with a cyclic factor-group of order 4) and where Z is cyclic of order 6. Again this counterexample is typical. Theorem 8. If $G=LA$, where L has a cyclic normal subgroup of prime order with a cyclic factor-group of order 4, and where A is abelian, and if G is not soluble, then G has a subgroup H that can be mapped homomorphically onto the S_5. *K. A. Hirsch* (London).

Schenkman, Eugene. On the tower theorem for finite groups. Pacific J. Math. **5** (1955), 995–998.

Let G be a group with trivial centre, and $A(G)$ its automorphism group, where G is identified with its group of inner automorphisms. Wielandt [Math. Z. **45** (1939), 209–244] proved that if G is finite $A^{n+1}(G)=A^n(G)$ for large enough n, and the author [Amer. J. Math. **73** (1951), 453–474; MR **13**, 103] has proved a similar theorem for Lie algebras. Here he shows that the Lie algebra methods cannot be used to simplify the proof for the case of groups. For, if H^∞ is the intersection of the terms of the lower central series of H, it is not generally true that if H is subinvariant H^∞ is normal, even if G is soluble and the centralizer of H in G is 1. It is true, however, that if the centralizer of H is 1 and H is subinvariant then the centralizer of H^∞ is contained in H^∞. *Graham Higman.*

Calame, André; et Piccard, Sophie. Les relations caractéristiques des bases du groupe symétrique. C. R. Acad. Sci. Paris **240** (1955), 2477–2478.

The symmetric group S_6 has 114480 bases consisting of two generators, which may be transformed by inner automorphisms into 7 types invariant under outer automorphisms and 78 pairs interchanged by outer automorphisms. By means of reversible functional relations connecting pairs of bases, the number of types is cut to five, of which the last two are equivalent under outer automorphism. For each of the four fundamental types the two basis elements are connected by 5 relations, known in at least one case to be independent. E. H. Moore's defining relations for S_n, $n\geq 4$, of the form $a^n=b^2=(ba)^{n-1}=(baba^{-1})^3=(ba^iba^{-i})^2=1$ for $i=2, 3, \cdots, n-2$ are found not to be independent. *J. S. Frame.*

Azleckiĭ, S. P. On commutator-group isomorphism of subgroups of a finite group. Ural. Politehn. Inst. Trudy. **51** (1954), 86–91. (Russian)

Two subgroups H and K of a finite group G are said to be commutator-group isomorphic (KIP, the initials of the original Russian phrase) if there exists an isomorphism α of H onto K such that $\alpha(h)h^{-1}\in G'$, the commutator subgroup of G. The KIP relation decomposes G into equivalence classes with a refinement which is the decomposition into equivalence classes of conjugate subgroups. The cross-cut of all the members of a KIP class is normal in G, and in an abelian or hamiltonian group, all KIP classes are unitary. If a finite group can be expressed in two ways as a product of Sylow subgroups, then the Sylow subgroups involved corresponding to the same prime power are KIP if not identical. For a non-special group G [Kurosch, Theory of groups, Gostehizdat, Moscow-Leningrad, 1944, p. 215; MR **9**, 267; **15**, 681], the author determines characterizing conditions that G have precisely one non-unitary KIP class. These conditions include that G is of order p^aq where p and q are primes, that $q\equiv 1 \bmod p$ and that various other congruences are satisfied. For special groups, the characterizing conditions are that G is neither abelian or hamiltonian and is of order p^a. *F. Haimo* (St. Louis, Mo.).

Edge, W. L. The conjugate classes of the cubic surface group in an orthogonal representation. Proc. Roy. Soc. London. Ser. A. **233** (1955), 126–146.

The cubic surface group of order 51840, whose classes and characters were previously obtained by Frame [Ann. Mat. Pura Appl. (4) **32** (1951), 83–119; MR **13**, 817], is studied geometrically using its representation as the group of 5×5 orthogonal matrices of determinant 1 over GF(3). For such a matrix M the null spaces of $M-I$ and $M+I$ are denoted by S_+ and S_- and have even and odd dimension, respectively. According as the join Σ of S_+ and S_- is the whole projective 4-space α, a plane, or a point, the matrix M is said to be of category A, B, or C. Forty "m points" of α lie on the invariant quadric ω defined by $x'x\equiv 0 \pmod 3$, 36 "f points" have $x'x\equiv -1$, and 45 "h points" have $x'x\equiv 1$. The orthogonal matrices M are classified into 21 types according to the relationships between S_+, S_- and ω, and a further refinement produces the 25 classes of conjugates for the group by geometrical considerations. For example, the classes $A_3(h, H)$ and $A_3(f, F)$ contain respectively 45 and 36 involutions determined by an h or f point and its polar solid. The author did not mention the interesting fact that the subgroup K of monomial matrices is of index 27 and corresponds to the subgroup leaving fixed one of the 27 lines L on the cubic surface. Sixteen of the classes are represented in this subgroup. The matrices having four 0's in just one row and four in one column form a double coset of K containing 10 cosets that take L into an intersecting line and contain representatives of all the other classes. Finally the remaining matrices having five 0's, one in each row and column, form a double coset containing 16 cosets that take L into a line skew to L. The author illustrates the advantages of studying the cubic surface group by means of the geometry in α, but makes no attempt to go beyond the classification into conjugate sets to obtain the group characters, which was the main purpose of Frame's paper. *J. S. Frame.*

Asano, Keizo. Einfacher Beweis eines Brauerschen Satzes über Gruppencharaktere. Proc. Japan Acad. **31** (1955), 501–503.

A simple proof is given for some results of the reviewer [Ann. of Math. (2) **48** (1947), 502–514; **57** (1953), 357–377; MR **8**, 503, **14**, 844]. Actually, this proof is almost identical with one given recently by the reviewer and John Tate [ibid. **62** (1955), 1–7; MR **16**, 1087]. Asano's paper originated at the same time as the last paper, and his work was completely independent. *R. Brauer.*

Osima, Masaru. On blocks of characters of the symmetric group. Proc. Japan Acad. **31** (1955), 131–134.

Two irreducible representations of the symmetric group belong to the same p-block, if and only if their Young diagrams have the same p-core. This was conjectured by T. Nakayama and first proved by the reviewer

[Trans. Roy. Soc. Canada. Sect. III. (3) 41(1947), 11–19; MR 10, 678] and G. de B. Robinson [ibid. 41 (1947), 20–25; MR 10, 678]. The author gives a new proof.

R. Brauer (Cambridge, Mass.).

Kaufman, A. M. Associative systems with an ideally solvable series of length two. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 67–93. (Russian)

Let S be a finite semigroup ("semigroup" means "set with a binary associative operation") containing an ideal T such that both T and the difference semigroup $S-T$ are generated by a single element. [For ideals and difference semigroups, see Rees, Proc. Cambridge Philos. Soc. 36 (1940), 387–400; MR 2, 127.] The author classifies all such semigroups in terms of certain positive integer invariants. For the algebraic details, which are both long and complicated, the reader is referred to the original paper.

E. Hewitt (Seattle, Wash.).

Kontorovič, P. G. On the theory of semigroups in a group. Kazan. Gos. Univ. Uč. Zap. 114 (1954), no. 8, 35–43. (Russian)

The author develops the structure of semigroups in a torsion-free group G. The first part is devoted to general structure properties of a semigroup S in terms of the following concepts: (i) the group kernel of S, $K(S)=S\cap S^{-1}$; (ii) the pure subsemigroup $S^*=S-S\cap S^{-1}$; (iii) the idealizer $u(S)$=the largest semigroup of G in which S is an ideal; (iv) the group closure $\bar{S}$=the group generated by S; (v) the normalizer $N(S)=\{x \in G \mid xSx^{-1}$ and $x^{-1}Sx \subset S\}$; (vi) the isolator $I(S)$=the minimal isolated semigroup of G containing S (a semigroup T is called isolated in G if for every $g \in G$ such that $g^n \in T$, for some integer n, it follows that $g \in T$).

In the second part, the author defines semigroups A and B to be a non-trivial covering pair for G if $A \cup B=G$ and neither A nor B is 1 or G. A semigroup A is called a linear semigroup if A and A^{-1} form a non-trivial covering pair for G. Along with other structure properties it is shown that one of the components of a covering pair must be a linear semigroup.

A semigroup S is said to be self-invariant if it is invariant in its group closure $\bar{S}$. A subsemigroup T of a semigroup S is called S-convex if the complement $\pi=S-T$ is an ideal of S. The last part of the paper considers the structure of ideals in self-invariant semigroups with respect to the prime ideals and extends previous results [Dokl. Akad. Nauk SSSR (N.S.) 93 (1953), 229–231; MR 15, 681].

L. J. Paige.

Stolt, Bengt. Über eine besondere Halbgruppe. Ark. Mat. 3 (1956), 275–286.

The author considers semigroups satisfying the following additional axiom: "For two arbitrary elements b, c there exists a unique element a satisfying $ab=c$." A structure of semigroups satisfying this axiom and the equivalence of semigroups defined by Prachar [Akad. Wiss. Wien. S.-B. IIa. 155 (1947), 97–102; MR 9, 491; 10, 855; 14, 1277] and Suschkewitsch [Math. Ann. 99 (1928), 30–50] are obtained.

L. J. Paige.

Lyapin, E. S. Canonical form of elements of an associative system given by defining relations. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 45–54. (Russian)

The author first sketches the theory of semigroups (sets with a binary associative operation defined by generators and relations. He then considers the semigroup P with generators u, v, e and the relations $ue=eu=u$, $ve=ev=v$, $e^2=e$, $vu=e$. Clearly every element of P has the form $u^\alpha v^\beta$ ($\alpha, \beta=0, 1, 2, \cdots$). It is proved that $u^\alpha v^\beta=u^\gamma v^\delta$ only if $\alpha=\gamma$ and $\beta=\delta$. P has only the trivial automorphism, and only one anti-automorphism, given by $u^\alpha v^\beta \to u^\gamma v^\alpha$.

E. Hewitt (Seattle, Wash.).

Lyapin, E. S. Increasing elements of associative systems. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 55–65. (Russian)

Notation and terminology are as in the preceding review. Let A be a semigroup. If $x \in A$, NCA, $N \neq A$, and $xN=A(Nx=A)$, then x is said to be a left (right) increasing element of A. Clearly if A is finite, it has no increasing elements, and if A has left (right) cancellation, then it has no left (right) increasing elements. A left increasing element cannot be right increasing. A is the union of 3 disjoint subsemigroups: the left increasing elements, the right increasing elements, and all other elements. The subsemigroup P of the preceding review contains both left and right increasing elements: if $\lambda>0$, then $v^\lambda(u^\lambda P)=P=(Pv^\lambda)u^\lambda$. Finally, let A be a semigroup with unit e. If x is a left increasing element in A, then there is an element $x' \in A$ such that the subsemigroup generated by $\{x', x, e\}$ is isomorphic to P ($x' \to u$, $x \to v$). Conversely, if A contains a subsemigroup generated by $\{x, y, e\}$ that is isomorphic in this way to P, then x is right increasing and y is left increasing.

E. Hewitt.

Vagner, V. V. Representation of ordered semigroups. Mat. Sb. N.S. 38(80) (1956), 203–240. (Russian)

Let A be a set. A single-valued function ϱ with domain and range subsets of A (proper or improper) is called a partial transformation of A. If the domain of ϱ is A, then ϱ is called a transformation of A. Under ordinary iteration, the set $\mathfrak{F}(A \times A)$ of all partial transformations of A forms a semigroup. The paper under review seeks to determine all representations of an arbitrary semigroup in semigroups of partial transformations, and in particular such representations of an ordered semigroup that preserve the ordering.

The first 2 §§ take up abstract closure operations (which seem to be unnecessary for later developments) and general properties of partial transformations. For example, it is shown that every semigroup of partial transformations of a set A is isomorphic to a semigroup of transformations of $A \cup \{c\}$, where c is a new point adjoined to A. The 3rd § lists a large number of formal properties of and equalities in semigroups.

Representations of a semigroup G in $\mathfrak{F}(A \times A)$ are taken up in § 4. A mapping P of G into $\mathfrak{F}(A \times A)$ is a representation if $P(g_1 g_2)=P(g_1) \circ P(g_2)$ for all $g_1, g_2 \in G$. Let Γ denote the set of all $P(g)$ ($g \in G$). Let τ be the binary relation $\mathbf{U}_{\gamma \in \Gamma} \gamma$ and let A_1 be the domain of τ. The weight of Γ is the least cardinal number of a subset α of A such that $A_1 \subset \alpha \cup \tau(\alpha)$. The author shows that every representation P of G in $\mathfrak{F}(A \times A)$ of weight w is the sum in a certain sense of w representations P_ι of G in $\mathfrak{F}(A \times A)$, where each P_ι has weight 1. Let G be a semigroup and G^* the semigroup obtained from G by adding a unit element. Then the transformations λ_g (right translations) of G^*, where $\lambda_g(h)=hg$ for sall $h \in G^*$, faithfully anti-represent G. The author shows that every representation of G in $\mathfrak{F}(A \times A)$ is obtainable as a homomorphic image of the product of w representations of G by right translations in G^*.

In § 5, the author takes up partial orderings in G that

are compatible with representations, as follows. Let σ be a quasi-order, that is, a reflexive and transitive binary relation, in G. Given a representation P of G in $\mathfrak{F}(A\times A)$, the binary relation ζ_P in G is defined as

$$\zeta_P=\{(g_1, g_2): P(g_1)\subset P(g_2)\}.$$

The quasi-order σ is said to be fundamentally representable if there is a P such that $\zeta_P=\sigma$. The set of all fundamentally representable σ's is studied in great detail, although it seems to the reviewer that no satis factory characterization of the semigroups admitting such quasi-orders is given. The paper closes with a description of all fundamentally representable quasi-orders for semigroups that are generated by a single element.

E. Hewitt (Seattle, Wash.).

Vorob'ev, N. N. On symmetric associative systems. Leningrad. Gos. Ped. Inst. Uč. Zap. **89** (1953), 161–166. (Russian)

Let M be a non-void set, with cardinal number m. Let C_m denote the set of all single-valued mappings of M into or onto itself that leave all but a finite number of points of M fixed. Under iteration, C_m forms a semigroup, called by the author a symmetric semigroup. The case $1\leqq m<\aleph_0$ has been studied earlier [Dokl. Akad. SSSR (N.S.) **58** (1947), 1877–1879; MR **9**, 330]. For $p\in C_m$, the cardinal number of the set $\{x: x\in M, x\neq p(y) \text{ for any } y\in M\}$ is called the defect of p. Obviously the defect is always finite. A subsemigroup A of C_m is called normal if for p, $q\in C_m$ and $a\in A$, one has $paq\in A$ if and only if $pq\in A$. [See Lyapin, Mat. Sb. N.S. **20(62)** (1947), 497–515; MR **9**, 134.] Theorem. For $m=2$, 3 and $m\geqq 5$, C_m has exactly 4 normal subsemigroups: C_m, the symmetric subgroup S_m of C_m, the alternating subgroup of S_m, and the identity. For $m=4$, there is in addition the Klein 4-group in S_4. A non-void subset T of C_m is called an ideal if $(C_mT)\cup(TC_m)\subset T$. Theorem. The ideals of C_m are exactly the sets $T_k=\{p: p\in C_m, \text{ defect of } p\geqq k\}$ $(k=0, 1, 2, \cdots)$.

E. Hewitt (Seattle, Wash.).

Bruck, R. H.; and Paige, Lowell J. Loops whose inner mappings are automorphisms. Ann. of Math. (2) **63** (1956), 308–323.

Sind $a, x, \cdots$ Elemente einer Loop G und die Rechts- und Links-Translationen $R(x)$, $L(x)$ definiert durch $ax=aR(x)$, $xa=aL(x)$, so erzeugt die Menge aller $R(x)$, $L(x)$ eine Gruppe G^*. Die Elemente U aus G^*, für die $1U=1$ ist, bilden eine Untergruppe J^* von G^*, die sogenannte innere Abbildungsgruppe; sie ist im allgemeinen keine Gruppe von Automorphismen von G. Verff. studieren diejenigen Loops, für die J^* eine Gruppe von Automorphismen von G ist, d.h. bei denen jedes Element U von J^* ein Automorphismus von G ist. Verff. bezeichnen diese Loops als A-Loops und bestimmt ihre Struktur ausführlich: Jede A-Loop ist potenzassoziativ und potenzkommutativ. Ferner folgt aus $x(yz)=(xy)z$ stets $x^m(y^nz^k)=(x^my^n)z^k)$, aus $xy=yx$ folgt stets $x^my^n=y^nx^m$ für alle ganzen, m, n, k. — Eine Loop heisst diassoziativ, wenn irgend zwei Elemente eine Gruppe erzeugen; eine Loop heisst Moufang-Loop, wenn die Identität $x(y\cdot zy)=(xy\cdot z)y$ besteht. In einer A-Loop ist die Diassoziativität äquivalent mit der Alternativität. In einer diassoziativen A-Loop erzeugen drei Elemente a, b, c mit $a(bc)=(ab)c$ eine assoziative Unterloop. Eine diassoziative A-Loop ist genau dann Moufang-Loop, wenn die Identität besteht $R(x)R(y)R(x)R(xyx)^{-1}=1$. Eine kommutative diassoziative A-Loop, in der die Ordnung jedes Elementes Potenz einer ungeraden Primzahl ist, ist Moufang-Loop, wenn sie im Sinne von R. H. Bruck [Trans. Amer. Math. Soc. **60** (1946), 245–354; MR **8**, 134] zentral nilpotent ist. — Verff. untersuchen weiter Isotope und Holomorphe von A-Loops. Insbesondere gilt, dass jede zu einer A-Loop G isotope Loop genau dann ihrerseits eine A-Loop ist, wenn $G'\subseteq N$, wo N der Kern von G ist und G' der Durchschnitt aller normalen Unterloops K aus G, so dass G/K eine abelsche Gruppe ist. Abschliessend geben Verff. einige Verfahren zur Konstruktion von A-Loops, insbesondere solchen, die keine kommutative Moufang-Loops sind.

R. Moufang (Frankfurt a.M.).

Oganesyan, V. A. Invariant and normal subsystems of a symmetric system of partial substitutions. Akad. Nauk Armyan. SSR. Dokl. **21** (1955), 49–56. (Russian. Armenian summary)

Let N be the set of integers $\{1, 2, \cdots, n\}$. If U and V are subsets of N containing k elements $(k\leqq n)$, a biunique mapping t of U upon V is a partial substitution of length k of the set N. The author defines the product xy of partial substitutions x and y to be the partial substitution $t=xy$ mapping all those and only those elements $a\in N$ for which $t(a)=y[x(a)]$.

A set of partial substitutions R is called a system of partial substitutions if it is closed under multiplication and the inverse of every $r\in R$ is in R. A subsystem K of a system R of partial substitutions is called normal if for all x, $y\in R$, $k\in K$, xky and xy are both in K or both are not in K. Defining Σ_n to be the symmetric system of all partial substitutions of N, all invariant subsystems (i.e., $sKs^{-1}\subseteq K$ for all $s\in\Sigma_n$ of length n) and normal subsystems of Σ_n are determined.

L. J. Paige.

See also: Iwasawa, p. 946; Stoka, p. 955; Fort, p. 994.

THEORY OF NUMBERS

★**Suškevič, A. K. Teoriya čisel. Èlementarnyĭ kurs. [Theory of numbers. An elementary course.]** Izdat. Har'kov. Gos. Univ., Kharkov, 1954. 204 pp. 5.20 rubles.

This text is a revision of the 1936 Ukrainian edition. The material covered is the same except that the chapter on quadratic forms has been reduced in content and a chapter on Russian number theorists has been added. The presentation is entirely elementary and there are many problems at the end of each chapter. The chapters are entitled: Divisibility, Euclid's algorithm and continued fractions, Congruences, Primitive roots and indices, Quadratic forms, Russian and Soviet works in number theory. The last chapter begins with Euler and includes a good deal of Chebyshev's original work on the distribution of primes. The works of Zolotarev, Voronoĭ, Vinogradov, and Gelfond are only briefly treated while others are barely mentioned. The conjecture of Golbach that every even number is the sum of two primes has apparently been deposited in a "memory hole." Instead Goldbach is made to say that every sufficiently large odd number is the sum of three primes, a theorem of Vinogradov. The work concludes with a table of powers of a primitive root and corresponding indices for prime moduli less than 100. This is taken, without reference, from the original table of M. Ostrogradsky (1838). This Russian number theorist, as

well as J. V. Uspensky, is not mentioned in the book.

The reviewer's curiosity is aroused by the bare statement on p. 143 that a table of indices for moduli <2000 exists. Such a table was indeed produced in 1955 by the SWAC in Los Angeles. No published table of this sort is known to the reviewer. *D. H. Lehmer.*

Alpár, László. Un problème de la théorie des nombres. Mat. Lapok **6** (1955), 309–322. (Hungarian. Russian and French summaries)

A number n is called multiply perfect if $\sigma(n)=kn$ for some integer k. The author determines all multiply perfect numbers of the form $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$. These are Euclid's numbers $2^{p-1}(2^p-1)$, where 2^p-1 is a prime and 120 and 672. *P. Erdős* (Haifa).

Golomb, Solomon W. Properties of consecutive integers. Nordisk Mat. Tidskr. **4** (1956), 24–29.

Generalizing the familiar theorem: There are arbitrarily long runs of composite numbers, the author first proves: If $A_1=a_1n+b_1$, $\cdots$, $A_k=a_kn+b_k$, where $(a_i, a_j)=1$ for $i\neq j$, then there are k consecutive integers $c+1$, $\cdots$, $c+k$ such that $c+j$ is in A_j for $1\leq j\leq k$. He next gives a simple proof of the following theorem of Wintner [The theory of measure in arithmetical semi-groups, Baltimore, Md., 1944, p. 14; MR **7**, 367]: There are arbitrarily long runs of consecutive integers $c+1$, $\cdots$, $c+k$, such that $\mu(c+1)=\cdots=\mu(c+k)=0$, where $\mu(n)$ is the Möbius function. It is observed that there are arbitrarily long sequences of non-kth-power-free integers. It is next proved that, for given M, there are runs such that

$$\phi(c+1)\equiv\cdots\equiv\phi(c+k)\equiv 0 \pmod{M},$$

where $\phi(n)$ is the Euler function. Like results are proved for the function $\sigma(n)$ and $d(n)$. The problem of whether for every k there exists a number c such that $d(c+1)=\cdots=d(c+k)$ is left open. *L. Carlitz* (Durham, N.C.).

Aczél, J.; et Zubrzycki, S. Sur un problème de la théorie des nombres lié à la distribution binomiale. Colloq. Math. **4** (1956), 56–67.

The rule for finding the largest term in the binomial distribution $\binom{n}{m}p^m(1-p)^{n-m}$ suggests the following definition. Let $0<p<1$; then p has the property Π if $p<\frac{1}{2}$ and there exist two integers a, n such that $a-\frac{1}{2}<np<a-p$, or $p<\frac{1}{2}$ and $a-p<np<a-\frac{1}{2}$. The main results of the paper are the following. 1) A necessary and sufficient condition that the irreducible fraction $p=i/j$ have the property Π is that $(i+1)/j<\frac{1}{2}$ for $p<\frac{1}{2}$ and $(i-1)/j>\frac{1}{2}$ for $p>\frac{1}{2}$. 2) All irrational numbers p in the interval $0<p<1$ have the property Π. 3) An explicit determination of the smallest n for a given p that has the property Π. *L. Carlitz* (Durham, N.C.).

Paradine, C. G. Farey series and Stern series. Math. Gaz. **40** (1956), 37–39.

The author considers a series of fractions obtained from the so-called Brocot series [Lucas, Théorie des nombres, t. 1, Gauthier-Villars, Paris, 1891, p. 470] by writing the latter double series as a single series after omitting all zero terms after the first, thus

$$\tfrac{0}{1}, \tfrac{1}{1}, \tfrac{1}{2}, \tfrac{1}{1}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{2}{3}, \tfrac{3}{5}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{5}{8}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}, \cdots.$$

A continued-fraction method is given for determining the nth fraction in the series and conversely a method for locating the first appearance of a given fraction. These methods are closely connected with those concerning the original Brocot or Stern series [cf. Lehmer, Amer. Math. Monthly **36** (1929), 59–67]. *D. H. Lehmer.*

Carlitz, L. The coefficients of $\sinh x/\sin x$. Math. Mag. **29** (1956), 193–197.

Put $\sinh x/\sin x=\sum_{m=0}^{\infty}\beta_{2m}x^{2m}/(2m)!$, so that the β's are rational numbers. In seeking a theorem of the Staudt-Clausen type for β_{2m} the author obtains the following result. The denominator of β_{2m} contains only primes $p\equiv 3 \pmod 4$. Let $p^r\mid 2m+1$, $p^{r+1}\nmid 2m+1$ and define u, v by means of $2m+1\equiv u+vp \pmod{p^2-1}$, $0\leq u\leq p-1$, $0\leq v\leq p-1$. Then

$$(2m+1)\beta_{2m}\equiv 1-\sum(-1)^s\binom{2m+1}{s(p-1)}\left(\frac{1}{p}-1\right) \pmod{p^r},$$

where $0<s(p-1)<2m+1$. If $u+v\geq p$, then the denominator of β_{2m} is divisible by exactly p^{r+1}; if $u+v<p$ the denominator of β_{2m} is divisible by p^r at most. Like results hold for the coefficients in the expansion of

$$(e^{\omega x}-1)/(e^x-1),$$

where ω is a primitive nth root of unity.

A. L. Whiteman (Los Angeles, Calif.).

Rényi, Alfred. On the density of certain sequences of integers. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 157–162.

If A is a set of positive integers and if $\lim_{x\to\infty}A(x)/x=d$, then d is called the density of A. Let $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ be the prime decomposition of n and consider the set A_k of integers for which $a_1-1+\cdots+a_r-1=k$. The author shows that A_k has density d_k where

$$\sum_{k=0}^{\infty}d_kz^k=\prod_p(1-p^{-1})(1-(p-z)^{-1}),$$

where the product is extended over all primes. [There is a misprint in the formula at the bottom of page 157.]

H. B. Mann (Columbus, Ohio).

Kac, M. A remark on the preceding paper by A. Rényi. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 163–165.

The author gives a probabilistic proof of the theorem of Rényi stated in the paper reviewed above.

H. B. Mann (Columbus, Ohio).

Fjellstedt, Lars. On a class of Diophantine equations. Ark. Mat. **3** (1956), 223–227.

The author applies to the equation

$$\sum_{i\leq u}(x_i+a_i)^2+c=x\prod_{i\leq u}x_i\neq 0 \quad (c\geq 0)$$

the methods of Hurwitz [Math. Werke, Bd. II, Birkhäuser, Basel, 1933, pp. 410–421] which have been applied recently by other authors to so many problems [Barnes, J. London Math. Soc. **28** (1953), 242–244; MR **14**, 725; Mills, Pacific J. Math. **3** (1953), 209–220; Proc. Amer. Math. Soc. **5** (1954), 473–475; MR **14**, 950; **16**, 13]. The equation has a solution x_1, $\cdots$, x_4 for only a finite number of x and these can be obtained from only a finite number of fundamental solutions in the now familiar way. The case $c<0$ is said to require more complication in detail and will be the subject of another paper.

J. W. S. Cassels (Cambridge, England).

Aigner, Alexander. Unmöglichkeitskernzahlen der kubischen Fermatgleichung mit Primfaktoren der Art $3n+1$. J. Reine Angew. Math. **195** (1955), 175–179 (1956).

The author continues his investigation of quadratic fields $K(\sqrt{m})$ in which $x^3+y^3+z^3=0$ is insoluble [same J. **195** (1956), 3–17; MR **17**, 464], this time with particular reference to the m divisible by several primes for which 2 is a cubic residue. [Note that the correct English translation of "Diese Gleichung ist in Prinzip möglich" is "I have so far been unable to disprove the existence of solutions of this equation".] *J. W. S. Cassels.*

Grebenyuk, D. G. On the representation of certain types of integers in the form of a sum of a definite number of cubes. Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 2, 3–6. (Russian. Uzbek summary)

The author notes that if the numbers $d_1, \cdots, d_m, \delta_1, \cdots, \delta_m, d, \delta$ satisfy the equations: (1) $\delta_1-d_1=\delta-d$, (2) $\delta_{i+1}-d_{i+1}=d_i+\delta_i$, (3) $d_m+\delta_m=d+\delta$, then (4) $d\delta=\sum d_i\delta_i$.

For fixed m, n he attempts to solve the diophantine equation (5) $n=\sum A_i^3$ by putting (6) $n=d\delta$, $A_i^3=d_i\delta_i$. On eliminating d, δ from (1), (3), (6) we have

$$4n=(d_m+\delta_m)^2-(\delta_1-d_1)^2.$$

Also from (6) we have

$$\begin{aligned} A_i &= l_i'''l_i^{\mathrm{IV}}(\Delta_i)^2\varepsilon_i'\varepsilon_i'' \\ \delta_i &= (l_i''')^2l_i^{\mathrm{IV}}(\varepsilon_i')^3\Delta_i^3 \\ d_i &= l_i'''(l_i^{\mathrm{IV}})^2(\varepsilon_i'')^3\Delta_i^3 \end{aligned} \tag{8}$$

(in the author's notation) for integers l_i''', l_i^{IV}, ε_i', ε_i'', Δ_i. If these integers can be found such that (8) satisfies (2), then we have a representation (5) of the number n given by (7).

The only special case discussed is $l_{i+1}'''=l_i'''+a$ $(1\leq i<m)$ $l_i^{\mathrm{IV}}=a$, $\varepsilon_i'=\varepsilon_i''=\Delta_i=1$ which, effectively, gives the sum of the series

$$(l_1''')^3+(l_1'''+a)^3+\cdots+(l_1'''+(m-1)a)^3.$$

J. W. S. Cassels (Cambridge, England).

Mycielski, Jan. On powers. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 129–132.

Denote by P the sequence of integers which are not powers. Clearly every integer can be uniquely represented in the form

$$P_1^{P_2}\cdot^{\cdot^{\cdot^{P_s}}}.$$

The author obtains various asymptotic formulas for sums $\sum_{P<x}(\log P)^\alpha$, and $\sum_{P<x}P^\alpha$ and various other asymptotic formulas. *P. Erdős* (Haifa).

Riesel, Hans. A note on the prime numbers of the forms $N=(6a+1)2^{2n-1}-1$ and $M=(6a-1)2^{2n}-1$. Ark. Mat. **3** (1956), 245–253.

For the primality of the numbers N and M indicated in the title, there are tests of the type of the well-known Lucas test for Mersenne numbers [D. H. Lehmer, Ann. of Math. (2) **31** (1930), 419–448; B. W. Brewer, Duke Math. J. **18** (1951), 757–763; MR **13**, 208]. The author gives direct proofs for these tests, and gives a list of all primes of the form $(6a\pm1)2^e-1$ with $6a\pm1<56$ and $1<e<150$ (in the cases $6a\pm1=5$, 7 or 11, the range is $1\leq e\leq250$). The calculations were made on the Swedish electronic computer BESK, using the above-mentioned tests. *N. G. de Bruijn* (Amsterdam).

Springer, T. A. Quadratic forms over fields with a discrete valuation. II. Norms. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 238–246.

This paper uses notations of a previous paper of the same title [same Proc. **58** (1955), 352–362; MR **17**, 17]. First we use this opportunity to correct errors made by the reviewer in reporting the previous paper. That paper treated definite quadratic forms and used as the norm of x, $\|x\|=|f(x,x)|^{\frac{1}{2}}$. There M_i is the set of vectors of $x\in E$ satisfying $\|x\|\leq|\pi|^i$ and π is defined in the review cited above. With these corrected definitions the assertions given in the previous review are correct except that the characteristic of the field K is restricted to be different from 2.

In paper II the forms are merely restricted to be non-degenerate and, as before, are on E, a finite-dimensional vector space over a commutative field K which is complete under a discrete valuation. Here a more general norm is used in accordance with a general definition given in the previous paper, i.e. the norm on E of x is a function $x\to\|x\|$ of E into the non-negative real numbers such that

$$\|x\|=0 \text{ is equivalent to } x=0,$$

$$\|x+y\|\leq\max(\|x\|,\|y\|),$$

$$\|\lambda x\|=|\lambda|\cdot\|x\| \quad (x,y\in E,\ \lambda\in K).$$

Such a norm is called a majorant of f if

$$|f(x,x)|<\|x\|^2 \quad (x\in E).$$

A majorant will be called an f-norm if it is minimal in the following sense: if $x\to\|x\|'$ is a majorant of f with $\|x\|'\leq\|x\|$ for all $x\in E$, then $\|x\|'=\|x\|$. This paper is concerned with f-norms.

The author proves that if $x\to\|x\|$ is a majorant of f on E, then it is an f-norm if and only if for every non-zero x in E, there is a y in E such that

$$\|y\|<\|x\|, \quad |f(x+y,x+y)|=\|x+y\|^2.$$

He shows that an f-norm may be defined with reference to a direct sum $E=E_0+E_1+\cdots+E_p$ of orthogonal subspaces where f is definite on E_0 and E_i are two-dimensional. He also shows that if f is non-degenerate and, when the characteristic of K is 2, non-defective (that is, the bilinear form g associated with f is non-degenerate) and if $x\to\|x\|$ and $x\to\|x\|'$ are two f-norms on E, then there exists an orthogonal transformation u of f such that $\|x\|'=\|u(x)\|$. Conversely, if $x\to\|x\|$ is an f-norm and u an orthogonal transformation, then $x\to\|u(x)\|$ is an f-norm.

B. W. Jones (Boulder, Col.).

Ebel, Ilse. Analytische Bestimmung der Darstellungsanzahlen natürlicher Zahlen durch spezielle ternäre quadratische Formen mit Kongruenzbedingungen. Math. Z. **64** (1956), 217–228.

Let

$$A\left(n\,\middle|\,\begin{matrix}a_1 & a_2 & a_3\\ k_1 & k_2 & k_3\end{matrix}\right)$$

denote the number of solutions of the equation $n=n_1^2+n_2^2+n_3^2$ subject to the congruence conditions $n_i\equiv a_i \pmod{k_i}$ $(i=1,2,3)$. The author determines explicit formulas for

$$A\left(n\,\middle|\,\begin{matrix}0 & 0 & 1\\ 4 & 4 & 4\end{matrix}\right) \text{ and } A\left(n\,\middle|\,\begin{matrix}1 & 2 & 2\\ 4 & 4 & 4\end{matrix}\right).$$

As usual, this is done by equating coefficients in an

appropriate identity involving modular forms. The identities are gotten by the methods of H. Petersson [Abh. Math. Sem. Hamburg Univ. **8** (1930), 215–242]. The equation $n=3n_1^2+n_2^2+n_3^2$ (subject to similar congruence relations) is also discussed for the triple $(k_1, k_2, k_3)=(2, 6, 6)$.

T. M. Apostol (Pasadena, Calif.).

de Fraga Torrejón, Eduardo. Note on Fermat's Last Theorem. Las Ciencias **21** (1956), 5–13. (Spanish)

Varnavides, P. Note on a theorem of Roth. J. London Math. Soc. **30** (1955), 325–326.

Roth proved that if $1\leqq a_1\leqq a_2<\cdots<a_k\leqq x$ is a sequence of integers and $k>\delta x$, then if x is sufficiently large, the equation $2a_l=a_i+a_j$ is always solvable with $i\neq j$ [C. R. Acad. Sci. Paris **234** (1952), 388–390; MR **13**, 724]. The author proves that there exists a constant $c(\delta)$ so that the number of solutions of $2a_l=a_i+a_j$ is greater than $c(\delta)x\log x$.

P. Erdös (Jerusalem).

Stöhr, Alfred. Bemerkungen zur additiven Zahlentheorie. III. Vereinfachter Beweis eines Satzes von A. Brauer. J. Reine Angew. Math. **195** (1955), 172–174 (1956).

[For parts I–II see same J. **183** (1941), 168–174; **185** (1943), 56–62; MR **4**, 241; **5**, 201.] The theorem in question is the following: Let A, B be sets of non-negative integers. Let B contain 0 and 1 and let A have positive density α. For every m let $l(m)$ be the smallest number l such that m is the sum of l numbers of B. Assume that $\sup \sum_1^n l(m)=\lambda<\infty$. Then $C(n)/n>(1+(1-\alpha)^{1/\lambda})$, where $C=A+B$.

H. B. Mann (Columbus, Ohio).

Halberstam, H. Über additive zahlentheoretische Funktionen. J. Reine Angew. Math. **195** (1955), 210–214 (1956).

An announcement of results whose proofs have already appeared [J. London Math. Soc. **31** (1956), 1–14, 14–27; MR **17**, 461].

W. J. LeVeque (Ann Arbor, Mich.).

Delange, Hubert. Sur un théorème d'Erdös et Kac. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 130–144.

The author gives a new proof of the following theorem due to the reviewer [J. London Math. Soc. **30** (1955), 43–53; MR **16**, 569]: Let $f(m)$ be a real strongly additive function defined on the set of positive integers by the properties (i) $f(mn)=f(m)+f(n)$, $(m, n)=1$, (ii) $f(p^\alpha)=f(p)$, p prime and $\alpha=2, 3, \cdots$. Define $A_k(x)=\sum_{p<x}[f(p)]^k/p$ and assume that (iii) $A_2(x)\to\infty$ as $x\to\infty$, (iv) $f(p)=O(1)$. Then $\sum_{n\leqq x}[f(n)-A_1(x)]^q=(\mu_q+o(1))x[A_2(x)]^{q/2}$, where $\mu_q=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}t^q e^{-t^2/2}dt$. The author was the first to prove this theorem in the special case $f(p)=1$ [C. R. Acad. Sci. Paris **237** (1953), 542–544; MR **15**, 201], and in his present proof of the general result he gains, by introducing certain types of generating function, some simplicity and elegance in method, although he sacrifies the entirely elementary nature of the reviewer's proof.

The author also proves briefly that the theorem remains true even if (iv) is sharpened to (v) $f(p)=o([A_2(p)]^{\frac{1}{2}})$, but he appears to err in instancing the scope of (v), which does not cover $f(p)=o(\log p)$. Indeed, for fixed $\alpha>0$, $f(p)=(\log p)^\alpha$ leads to an altogether different kind of distribution.

H. Halberstam (Exeter).

Carlitz, L. A further note on Dedekind sums. Duke Math. J. **23** (1956), 219–223.

The functional equation

$$f_m(h, k; \tau)=(-1)^m\tau^{m-2}f_m(-k, h; -\tau^{-1})+\tau^{-1}(B+\tau B)^m,$$

derived in an earlier paper [Proc. Amer. Math. Soc. **5** (1954), 580–584; MR **16**, 14] is generalized to the form

$$f_m(h, K; \tau)=(-1)^m(k\tau-h)^{m-2}f_m(h_1, k_1; \tau_1)+f_m(h, k; \tau),$$

where the integers h, k, h_1, k_1, K and the complex numbers τ and τ_1 are related as follows: $(h, k)=1$, $hh'+kk'+1=0$, $\tau_1=(h'\tau+k')/(k\tau-h)$, $(h_1, k_1)=1$, $K=h'k_1-h_1k$. Specialization leads to the three-term relation for Dedekind sums recently discovered by Rademacher [Duke Math. J. **21** (1954), 391–397; MR **16**, 14]. *T. M. Apostol.*

Newman, Morris. A table of the coefficients of the powers of $\eta(\tau)$. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 204–216.

The numbers $p_r(n)$ defined by the equation

$$\prod_{n=1}^{\infty}(1-x^n)^r=\sum_{n=0}^{\infty}p_r(n)x^n \quad (r=1, 2, \cdots)$$

are tabulated for $1\leqq r\leqq 13$, $1\leqq n\leqq 800$; $r=14$, $1\leqq n\leqq 750$; $r=15$, $1\leqq n\leqq 500$; $r=16$, $1\leqq n\leqq 400$. The computations were performed with the help of the high speed electronic digital computer SEAC of the National Bureau of Standards at Washington, D.C. Two different recursion formulas were used, namely:

$$np_r(n)=\sum_{k=1}^{n}(rk+k-n)\lambda_k p_r(n-k)$$

and

$$np_r(n)=-r\sum_{k=1}^{n}\sigma(k)p_r(n-k),$$

where $\sigma(k)$ is the sum of the divisors of k, and

$$\lambda_k=\begin{cases}(-1)^s \text{ if } k=(3s^2\pm s)/2\\ 0 \text{ otherwise.}\end{cases}$$

T. M. Apostol (Pasadena, Calif.).

See also: van der Waerden, p. 931.

Algebraic Number Theory

Iwasawa, Kenkichi. A note on the group of units of an algebraic number field. J. Math. Pures Appl. (9) **35** (1956), 189–192.

Let K/k be a finite normal extension of an algebraic number field, with Galois group G. Let E be the group of units in K. The author shows first that $H^1(G, E)$ is isomorphic to the group of ambiguous principal ideals of K modulo principal ideals of k; he mentions that this is well-known in another notation. Let J be the group of K-idèles, I the group of K-ideals, U the kernel of the natural homomorphism of J onto I, and P the principal K-idèles. Let $\bar{J}=J/P$ and $\bar{I}=J/PU$ be the idèle-class and ideal-class groups. Using methods of Artin, Tate, and Chevalley [Chevalley, Class field theory, Nagoya Univ., 1954, MR **16**, 678] he proves that if K/k is everywhere unramified then $H^n(U/E)\cong H^{n+1}(E)$ for all n, and that the sequence

$$\cdots\to H^{n+1}(E)\to H^n(\bar{J})\to H^n(\bar{I})\to H^{n+2}(E)\to\cdots$$

is exact. It follows that $H^2(G, E)$ is isomorphic to the group of ambiguous ideal classes of K modulo the ideal classes represented by ideals of k. *G. Whaples.*

Popovici, Constantin P. Sur la détermination d'une base des entiers du corps relativement quadratique de Dirichlet. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 3 (1954), no. 4–5, 47–52. (Romanian. French and Russian summaries)

The field of Dirichlet is defined as the biquadratic field $R(i, \sqrt{\delta})$, obtained by adjoining to the rationals the imaginary unit and the square root of a square-free integer δ of $R(i)$. The integers of this field are of the form $\alpha+\beta\sqrt{\delta}$, $\alpha, \beta \in R(i)$ and satisfy an equation $t^2-2\alpha t+(\alpha^2-\beta^2\delta)=0$, whose coefficients have to be integers in $R(i)$. By imposing this condition and using many case distinctions, the well-known [see Hilbert, Math. Ann. **45** (1894), 309–340] result is obtained, that the integers of Dirichlet are of the form $\sigma+\tau\theta$, where σ and τ are Gaussian integers and θ has one of the values $\frac{1}{2}(1+\sqrt{\delta})$, $(1+\sqrt{\delta})/(1+i)$, $\frac{1}{2}(1+i\sqrt{\delta})$, or $\sqrt{\delta}$, according to the residue class (mod 4) of δ.

E. Grosswald (Philadelphia, Pa.).

Carlitz, L. A special symmetric equation in a finite field. Acta Math. Acad. Sci. Hungar. **6** (1955), 445–450. (Russian summary)

Let σ_r denote the rth elementary symmetric function of $x_1, \cdots, x_4$. Mordell [Canad. J. Math. **5** (1953), 433–438; MR **15**, 200] proved that the number of solutions of the congruence $a_1\sigma_1+\cdots+a_4\sigma_4\equiv a \pmod p$ is equal to $p^3+O(p^{3/2})$ except in certain excluded cases. The present author determines explicitly the number $M=M(a)$ of solutions of the equation $\sigma_3=a$, where $a, x_r \in GF(q)$, $q=p^n$ and p is an arbitrary prime. A particular case of the main result states that if q is odd, then

$$M(a)=q^3-2q \quad (a\neq 0,\ q\not\equiv 1 \pmod 3),$$

$$M(a)=q^3-2q+(-1)^{n/2}tq^{3/2} \quad (a\neq 0,\ q\equiv 1,\ p\equiv 2 \pmod 3),$$

where $t=-2$ or 1 according as a is or is not a cube of $GF(q)$.

A. L. Whiteman (Los Angeles, Calif.).

Šidlovskiĭ, A. B. On transcendentality of the values of a class of entire functions satisfying linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 35–37. (Russian)

This is one of a series of papers by the author [same Dokl. (N.S.) **96** (1954), 697–700; **100** (1955), 221–224; **103** (1955), 977–980; MR **16**, 117, 907; **17**, 466] extending and generalizing Siegel's work [Transcendental numbers, Princeton, 1949, p. 52; MR **11**, 330] on E-functions. The following theorems are announced: I. Let the E-function $f(z)$ be a solution of the linear differential equation $P_m(z)y^{(m)}+\cdots+P_1(z)y'+P_0(z)y=Q(z)$, whose coefficients $P_i(z)$ and $Q(z)$ are polynomials in z with algebraic numerical coefficients, and let α be an algebraic number different from 0 and the zeros of $P_l(z)$. Then the l numbers $f^{(i)}(\alpha)$, $i=0, 1, \cdots, l-1$, $1\leq l\leq m$, are algebraically independent if and only if the functions $f^{(i)}(z)$, $i=0, \cdots, l-1$ are algebraically independent over the field of rational functions of z. II. If a function f as in I is transcendental, and if $\alpha\neq 0$ is algebraic and not a zero of $P_m(z)$, then the numbers $f^{(i)}(\alpha)$, $i=0, 1, \cdots$ are transcendental and all zeros and all A-values of f and all its derivatives, except 0 and the zeros of $P_m(z)$ are transcendental for arbitrary algebraic A.

These theorems are very much easier to apply than Siegel's theorem, since Siegel's condition of normality has disappeared. It follows from II, for example, that every hypergeometric E-function [cf. Siegel, op. cit., p. 54] and each of its derivatives assumes a transcendental value for algebraic argument different from 0.

W. J. LeVeque.

Hasse, Helmut. Zetafunktion und L-Funktionen zu einem arithmetischen Funktionenkörper vom Fermatschen Typus. Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1954**, no. 4, 70 pp. (1955).

Let k be a number field, t an indeterminate over k, and K a finite algebraic extension of the rational function field $R=k(t)$. The field K is called arithmetical function field. The author defines a zeta function $\zeta_K(s)=\prod_{\mathfrak{p}} \zeta_{K\mathfrak{p}}(s)$, where $\mathfrak{p}$ ranges over the "arithmetic primes" of K (i.e. those places for which the image field $k\mathfrak{p}$ is finite and $t\mathfrak{p}$ is transcendental over $k\mathfrak{p}$); $\zeta_{K\mathfrak{p}}(s)$ denotes the well-known zeta function of the image field $K\mathfrak{p}$ regarded as a function field over a finite field as field of constants. In this paper, the author first gives the foundations of a general theory of arithmetic function fields and their zeta functions. Secondly, he considers the so-called (special) Fermat function fields, defined by $K=k(u_1, u_2)$ with $u_1^{m_1}+u_2^{m_2}=1$ (m_1, m_2 are positive integers); here t is to be $t=u_1^{m_1}=1-u_2^{m_2}$. Assume that k contains the m_ith roots of unity ($i=1, 2$). In these fields he proves a decomposition law for the arithmetic primes with respect to K/R. By means of this, he shows that $\zeta_K(s)$ splits into the product of $\zeta_R(s)=\zeta_k(s)\zeta_k(s-1)$ and another factor $L_K(s)$ which in turn splits into certain L-functions $L(\chi|s)$ corresponding to the characters $\chi\neq 1$ of the Galois group of K/R. It is proved that $L(\chi|s)^{-1}$ can be represented by means of a certain "Größencharakter" λ_χ (in the sense of Hecke) as the Euler product

$$L(\chi|s)^{-1}=\prod_{\mathfrak{p}}(1-\lambda_\chi(\mathfrak{p})\mathfrak{N}(\mathfrak{p})^{s-1}),$$

where $\mathfrak{p}$ ranges over all the primes of k not dividing the order $m(\chi)$ of χ. The conductor $\mathfrak{f}(\chi)$ of λ_χ divides $m(\chi)^2$. The author determines $\mathfrak{f}(\chi)$ in certain special cases. [See also A. Weil, Trans. Amer. Math. Soc. **73** (1952), 487–495; MR **14**, 452; M. Deuring, Nachr. Akad. Wiss. Göttingen. Math. Phys. Kl. IIa. **1953**, 85–94; MR **15**, 779; E. Lamprecht, Arch. Math. **6** (1955), 266–274; MR **17**, 350; see also the paper reviewed below.]

P. Roquette.

Lamprecht, Erich. Zetafunktionen symmetrisch-erzeugbarer algebraischer Funktionenkörper mehrerer Veränderlicher. Math. Z. **64** (1955), 47–71 (1956).

Let A be a finitely generated field and p the characteristic of A. Assume there is given a basis of transcendency $x=\{x_1, \cdots, x_n\}$ of A. The author defines a zeta-function $Z_A(s)$, depending on x, in the following recursive way [in the case $p=0$, $n=1$ see also the paper reviewed above]: If A is a finite field (i.e., $p>0$, $n=0$), put $Z_A(s)=(1-\mathfrak{N}(A)^{-s})^{-1}$, where $\mathfrak{N}(A)$ denotes the number of elements in A. If A is not finite, let A_* be the subfield of those elements in A, which are absolutely algebraic if $p=0$, and which are algebraic over the field generated by x_1 if $p>0$. Consider the places $\mathfrak{P}$ of A such that (i) $\mathfrak{P}$ induces in A_* a non-trivial place, and (ii) the images $x_i\mathfrak{P}$ ($1\leq i\leq n$ if $p=0$; $2\leq i\leq n$ if $p>0$) are algebraically independent over $A_*\mathfrak{P}$. By induction, assume the zeta-function $Z_{A\mathfrak{P}}(s)$ of the image field $A\mathfrak{P}$ with respect to the basis of transcendency $\{x_1\mathfrak{P}, \cdots, x_n\mathfrak{P}$ if $p=0$, and $x_2\mathfrak{P}, \cdots, x_n\mathfrak{P}$ if $p>0\}$ already defined. Then $Z_A(s)$ is defined by the product formula $Z_A(s)=\prod_{\mathfrak{P}} Z_{A\mathfrak{P}}(s)$.

The author considers mainly the case where A is the independent compositum of n fields K_i, each of degree of transcendency 1, and where x_i is a basis of transcendency for K_i. By means of general composition theorems (§ 1, 3) he shows that the properties of $Z_A(s)$ can be derived from those of the $Z_{K_i}(s)$. He discusses the following properties:

(a) convergence of the defining product formula; (b) rationality of $Z_A(s)$ as function of q^s if $p>0$ and q is the number of elements in the constant field of A; (c) representation of $Z_A(s)$ by means of L-functions with „Größencharaktere" if $p=0$; (d) distribution of poles and zeros if $p>0$; (e) functional equation if $p>0$.

In § 2, the author considers the case $p>0$, and the "geometrical" zetafunction $\zeta_A(z)$, defined with respect to a projective model M of A by

$$d \log \zeta_A(z)=\sum_r N^{(r)}(M)\, z^r dz/z,$$

here $N^{(r)}(M)$ denotes the number of points of M which are rational in the extension of degree r over the constant field of A [see Weil, Bull. Amer. Math. Soc. **55** (1949), 497–508; MR **10**, 592]. He again proves a general composition theorem, similar to that for the zetafunctions described above. From this it follows, if A is the independent compositum of n fields K_i, that $\zeta_A(q^{-s})=Z_A(s)$, if the choices of M and x, defining ζ and Z, are both adapted to the representation of A as the compositum of the K_i.

The author mentions that in general it is not true that $\zeta_A(q^{-s})=Z_A(s)$, even if M and x are related to each other in a natural way. He computes $\zeta_A(z)$ if M is defined by the inhomogeneous equation $y^p-y=a_1x_1^m+\cdots+a_nx_n^m$ $(m|p-1)$. [For the case $n=1$, see Davenport and Hasse, J. Reine Angew. Math. **172** (1934), 151–182.] Finally, the author gives necessary and sufficient conditions for A (if $n=2$) to be represented in two different ways as independent compositum of two fields. *P. Roquette.*

See also: Uchiyama, p. 937.

Geometry of Numbers, Diophantine Approximation

Ehrhart, Eugène. Sur les polygones et les ovales. C. R. Acad. Sci. Paris **242** (1956), 332–334.

In an earlier note [same C. R. **241** (1955), 686–689; MR **17**, 350], the author has shown that the area of a convex polygon Π, whose vertices are situated at points of the lattice with integral coordinates, is simply related to the number of such lattice points in Π. A similar result is obtained here for more general polygons. The note also contains a theorem giving bounds for the ratio in which the pericentre of an oval divides a chord passing through it. *J. H. H. Chalk* (London).

Ehrhart, Eugène. Sur les polygones croisés. C. R. Acad. Sci. Paris **242** (1956), 1570–1573.

This is a continuation of two previous notes [same C. R. **241** (1955), 686–689; MR **17**, 350; see also the paper reviewed above] on the relation between the area of a polygon Π, with vertices at points of the lattice with integral coordinates, and the number of such lattice points in Π. It concludes with some remarks on ovals, and contains a revised conjecture for non-central convex bodies (ovoids) [see footnote (4) on p. 484 of Ehrhart, ibid **240** (1955), 483–485; MR **16**, 574]. *J. H. H. Chalk.*

Sudan, Gabriel. Interprétation géométrique d'une certaine équation en nombre entiers. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 23–30. (Romanian. Russian and French summaries)

If $b>a>0$, $(a, b)=1$, then the diophantine equation $ax-by=z$ has at least one, and at most two solutions in integers x, y, z, satisfying $|x|<\sqrt{b}$, $|y|<\sqrt{b}$, $0<|z|<\sqrt{b}$, $(x, y)=1$ (the solutions (x, y, z) and $(-x, -y, -z)$ are not considered distinct). This theorem of Zeitz [Jber. Deutsch. Math. Verein. **42**, Abt. 2, 72 (1932), 110 (1933)] is proven by a geometrical argument, using Minkowski's theorem on convex figures. Furthermore, it is shown that if there is only one solution, this is a convergent of a/b, while if there are two solutions, at least one is a convergent. *E. Grosswald* (Philadelphia, Pa.).

Platonov, M. L. An estimate of the approximation of algebraic irrationalities by ratios of logarithms of increasing natural numbers. Irkutsk. Gos. Univ. Trudy. **8** (1953), no. 1, 53–62. (Russian)

Let α be an irrational algebraic number and let ε be positive. Then there is a $q_0>0$ such that for every pair of positive integers p and $q>q_0$ for which $\log p/\log q$ is irrational, the inequality

$$\left|\alpha-\frac{\log p}{\log q}\right|>\exp\left(-\log^2 q \log^{\delta+\varepsilon}\log q\right)$$

holds. The proof depends on a method described by A. O. Gelfond [Uspehi Mat. Nauk (N.S.) **4** (1949), no. 4(32), 19–49; MR **11**, 231]. *W. J. LeVeque* (Ann Arbor, Mich.).

Gorškov, D. S. Non-quadratic algebraic irrationalities decomposible into continued fractions with bounded sets of partial quotients. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 383–384. (Russian)

Let A be a positive integer containing a prime $p\equiv 3 \pmod 4$ to an odd power, and let m be an arbitrary real number with $m^2<A$. It is shown that the number $m+i(A-m^2)^{\frac{1}{2}}$ has a Hurwitz continued fraction [Acta Math. **11** (1888), 187–200] whose partial quotients are bounded in absolute value. *W. J. LeVeque.*

Descombes, Roger. Sur un problème d'approximation diophantienne. I, II. C. R. Acad. Sci. Paris **242** (1956), 1669–1672, 1782–1784.

Let ξ be a real irrational number, and η a real number, and put $c(\xi, \eta)=\limsup (v|v\xi-u-\eta|)^{-1}$, the limit being over all pairs of integers $v\neq 0$, u. J. W. S. Cassels [Math. Ann. **127** (1954), 288–304; MR **15**, 687] used an algorithm similar to a continued fraction to evaluate the smallest value of $c(\xi, \eta)$ and to show that it is an isolated value. In the present notes the author announces the results of further investigations based on Cassel's method. He defines equivalence of pairs (ξ, η), points out that equivalent pairs give the same value of c, and lists the values of $c(\xi, \eta)$ between the minimum and the smallest limit point, together with the corresponding ξ and η. The smallest limit is $366795/(773868-28547\sqrt{510})=2.839\cdots$. *W. J. LeVeque* (Ann Arbor, Mich.).

ANALYSIS

Have

★ **Strubecker, Karl. Einführung in die höhere Mathematik mit besonderer Berücksichtigung ihrer Anwendungen auf Geometrie, Physik, Naturwissenschaften und Technik. Bd. I. Grundlagen.** R. Oldenbourg Verlag, München, 1956. xv+821 pp. DM 36.00.

This outline of modern higher mathematics, planned for three volumes, is based on "lectures delivered especially to physicists, engineers, chemists and biologists", but it is also quite suitable for students wishing to specialize in mathematics. The present first volume deals chiefly (see preface) with I) real numbers; II) elementary algebraic functions, including polynomials, questions of

interpolation, affine and projective geometry and vector algebra of the plane; III) theory of limits and infinite series (real and complex); IV) elementary transcendental functions, slide rule, logarithmic paper. The proposed content of volumes two and three is not indicated. There are no exercises as such, but occasional passages are labelled (Aufgabe!). The style is everywhere unhurried and pleasant, with numerous illuminating remarks of an historical or linguistic nature. There is a bibliography at the end of each chapter. The book appears well adapted for lectures or for private study. *S. H. Gould.*

★**Newell, Homer E., Jr. Vector analysis.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. xi+216 pp. $5.50.

This is a book for American students in physics and engineering containing numerous problems. The contents of the chapters run as follows: 1) algebra of vectors; 2) survey of ordinary calculus; 3) the gradient and differentation of vectors; 4) divergence, curl, Stokes; 5) and 6) the operator ∇; 7) curvilinear coordinates (very elementary); 8) vector fields and potential theory. The remaining part of the book contains practical applications: 9) miscellaneous applications on forces, velocities etc.; 10) motions in space; 11) electromagnetic theory (classical). There is no treatment of more dimensions or tensor calculus. The principal criterion for the author has been that the text should be understandable and useful to the serious student. *J. A. Schouten* (Epe).

Stipanič, E. Une contribution au traitement méthodique des théorèmes de Rolle, de Lagrange et de Cauchy. Mathesis **65** (1956), 67–69.

Menger, Karl. What are variables and constants? Science **123** (1956), 547–548.

This deals with the different meanings of the term variable in applications of mathematics to the sciences, and the importance of distinguishing clearly between them. The discussion is similar to that in the author's textbook, "Calculus, a modern approach" [Ginn, New York, 1955; MR **16**, 575; **17**, 351]. *O. Frink.*

Velasco de Pando, Manuel. On some algebroid functions related with systematic integration of differential elements. Las Ciencias **17** (1952), 5–16. (Spanish)

Tandori, Károly. On the convergence of singular integrals. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 61–68. (Hungarian)

Hungarian version of a previous paper [Acta Sci Math. Szeged **15** (1954), 223–230; MR **16**, 1091].

★**Mandel'broĭt [Mandelbrojt], S. Primykayuščie ryady. Regulyarizaciya posledovatel'nosteĭ. Priloženiya. [Séries adhérentes. Régularisation des suites. Applications.]** Izdat. Inostrannoĭ Literatury, Moscow, 1955. 268 pp. (12 tables).

The original appeared in 1952 [Gauthier-Villars, Paris; MR **14**, 542]. The translation is straightforward except that "Schwarz's inequality" becomes "Bunyakosvkiĭ's inequality" and except for an error, in a footnote on the first page where "le premier indice d'une suite sera 1" is incorrectly translated into "the first term of a sequence will be one". At the end there is added a bibliography of papers concerning the subject published after 1952. There is an introduction by V. L. Gončarov.

Bajraktarević, M. Sur les itérées continues et leur application à la recherche des fonctions limites de certaines suites itérées. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 13–22.

Let $F(x)$ be continuous and strictly increasing on $[a, b]$ with either (1a) $F(x)>x$ on $a\leq x<b$ and $F(b)=b$, or (1b) $F(x)<x$ on $a<x\leq b$ and $F(a)=a$. For case (1a), Ward and Fuller [Bull. Amer. Math. Soc. **42** (1936), 393–396] defined the concept of continuous iterate of $F(x)$, and proved some theorems regarding such. These results are now extended to case (1b), and application made to the limit functions of certain sequences of iterate functions. *I. M. Sheffer* (University Park, Pa.).

Bajraktarević, Mahmud. Sur certaines suites itérées. Naučno Društvo NR Bosne i Hercegovine. Djela **4**, Odjeljenje Privredno-Tehničkih Nauka **1** (1954), 33 pp.

In a series of notes [C. R. Acad. Sci. Paris **236** (1953), 881–883; 988–989, 1125–1127; MR **14**, 760] and in an earlier article [Srpska Akad. Nauka Zb. Rad. **35**. Mat. Inst. **3** (1953), 61–74; MR **15**, 784] the author defined the iterated sequences with which he deals, and gave many of their properties. The present work is an elaboration of the papers appearing earlier, giving all details, and also extending the results. (The reviews cited above contain the important definitions and statement of some of the results.) The author also discusses and extends work of Ward and Fuller [Bull. Amer. Math. Soc. **42** (1936), 393–396] on the concept of continuous iteration of real functions. *I. M. Sheffer* (University Park, Pa.).

van der Corput, J. G.; and Heflinger, L. O. On the inequality of Mathieu. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 15–20.

Authors give another proof of the inequality

$$\sum_{n=1}^{\infty} n(n^2+x^2)^{-2}<1/(2x^2)$$

conjectured by E. Mathieu [Traité de physique mathématique, t. VI–VII, Gauthier-Villars, Paris, 1890] and proved by L. Berg [Math. Nachr. **7** (1952), 257–259; MR **14**, 731] while he shows that the proof of O. Emersleben [Math. Ann. 125 (1952), 165–171; MR **14**, 369] contains a fault which however can be corrected. The calculations of the authors are based on following theorems: 1. For the integers $a<b$ let $f^{1v}(x)$ exist in $[a, b]$ and be of definite sign in its subintervals (a_k, a_{k+1}) $(k=0, 1, \cdots, q;\ a_0=a,\ a_{q+1}=b)$: $f^{1v}(x)\leqq 0$ for (a_{m-1}, a_m) and $f^{1v}(x)\geqq 0$ for (a_m, a_{m+1}) where m takes I) all even, resp. II) all odd values $(a_{-1}=a,\ a_{q+2}=b)$ and q is A) even, B) odd. It is also assumed that

IA) $$32[f'(a)-f'(b)]>\sum_{k=1}^{q}(-1)^{k+1}f'''(a_k),$$

IB) $$32[f'(a)-f'(b)]>\sum_{k=1}^{q+1}(-1)^{k+1}f'''(a_k),$$

IIA) $$32[f'(a)-f'(b)]>\sum_{k=0}^{q+1}(-1)^{k}f'''(a_k),$$

IIB) $$32[f'(a)-f'(b)]>\sum_{k=0}^{q}(-1)^{k}f'''(a_k),$$

respectively. Then

$$\tfrac{1}{2}f(a)+\tfrac{1}{2}f(b)+\sum_{n=a+1}^{b-1}f(n)<\int_a^b f(x)dx.$$

2. $$\tfrac{1}{8}\int_a^b f''^{-}(x)dx<\tfrac{1}{2}f(a)+\tfrac{1}{2}f(b)+\sum_{n=a+1}^{b-1}f(n)-\int_a^b f(x)dx<\tfrac{1}{8}\int_a^b f''^{+}(x)dx$$

$(f''^{\pm}(x)=\frac{1}{2}[|f''(x)|\pm f''(x)]$, a, b are integers and $f''(x)$ exists in $[a, b]$). *J. Aczél* (Debrecen).

Kjellberg, Bo. A note on an inequality. Ark. Mat. 3 (1956), 293–294.

This note is a supplement to a much earlier one [Den 01. Skand. Matematikerkongres, København, 1946, Gjellerup, København, 1947, pp. 333–340; MR 8, 448] concerning an inequality of the type $J \leq K(J')^{\lambda}(J'')^{\mu}$, where J, J', J'' are integrals on $0<x<\infty$ of the form $\int x^{\alpha}|f(x)|^{\beta}dx$, for 3 pairs of exponents (α, β) related by a triangular condition. The author observes that a simple method, there indicated, yields also the extremal function and the best value of K. The latter has been discussed by various authors, in particular Levin [Dokl. Akad. Nauk SSSR (N.S.) 59 (1948), 635–638; MR 9, 415]. *L. C. Young.*

Manevič, D. V. Extrema of functions of a discrete variable. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. 11 (1953), 92–96. (Russian)

The author considers the problem of determining the maximum of $f(a+kh)$ as k varies over the sequence $k=0, 1, 2, \cdots$. The author states some necessary and sufficient conditions and considers a particular problem. *R. Bellman* (Santa Monica, Calif.).

Mallows, C. L. Note on the moment-problem for unimodal distributions when one or both terminals are known. Biometrika 43 (1956), 224–227.

Johnson and Rogers [Ann. Math. Statist. 22 (1951), 433–439; MR 13, 119] gave determinantal criteria for the moment problem on $(-\infty, \infty)$ with a unimodal distribution. The authors extend this to the moment problem on $(0, \infty)$ or (a, b). Several necessary conditions are written out explicitly. *R. P. Boas, Jr.*

See also: Aljančić, Bojanić et Tomić, p. 963; Izumi, p. 963; Kilpi, p. 989; Hasselmeier, p. 1008.

Theory of Sets, Functions of Real Variables

Thiele, Ernst-Jochen. Ein axiomatisches System der Mengenlehre nach Zermelo und Fraenkel. Z. Math. Logik Grundlagen Math. 1 (1955), 173–195.

This paper presents a formalization of the set theory of Zermelo and Fraenkel [J. Fraenkel, Reine Angew. Math. 155 (1926), 129–158] within the first-order predicate calculus with a single primitive relation ϵ, and a development of the theory of cardinal and ordinal numbers within the formalized theory. In addition to the Aussonderungs- and Ersetzungs-axiom schemes:

$$(\forall m_0 \exists m \forall x(x \epsilon m \leftrightarrow x \epsilon m_0 \wedge H(x)),$$

and

$$\forall x(x \epsilon m \rightarrow \exists y H(x, y)) \rightarrow \exists a \forall x(x \epsilon m \rightarrow \exists y(y \epsilon a \wedge H(x, y)),$$

for any statement $H(x)$ not containing m free and any statement $H(x, y)$ containing x and y free but not containing a free) there are axioms guaranteeing the existence of the power set of any set, the set of all elements of elements of any given set, and for any two given sets the set containing just these sets, as well as the extensionality axiom, the axiom of infinity and the axiom of choice. All but the last section of the paper makes no use of the Ersetzungs-axiom scheme.

The theory of cardinal and ordinal numbers differs from the usual theory because of the limitation of the Aussonderungs-axiom scheme. This axiom scheme permits in the case of cardinal-number theory, for example, only the formation of the set of elements of a given set m_0 which can be mapped one-one onto a given set. Therefore, in place of "y is the cardinal number of the set m" appears "y is the cardinal number of the set m relative to the set p" defined formally:

$$\overset{m}{m} \leq p \wedge \forall x(x \epsilon y \leftrightarrow x \epsilon Pp \wedge x \overset{m}{\sim} m),$$

where $\overset{m}{m} \leq p$ means that there is a one-one mapping of m into p, $x \overset{m}{\sim} m$ means that there is a one-one mapping of x onto m, and where Pp is the power set of p. Similarly, in place of the usual ordinal numbers appear ordinal numbers relative to a given set.

The last section is devoted to a discussion of the strength of the set theory. It is proven that for every ordinal number α there is a set with cardinality $\aleph_\alpha$. Further there is proven that there exists a set of every cardinality smaller than that of the first non-denumerable unerreichbar set. A related theorem for ordinal numbers is also proven. *P. C. Gilmore* (University Park, Pa.).

Kinna, W.; und Wagner, K. Über eine Abschwächung des Auswahlpostulates. Fund. Math. 42 (1955), 75–82.

If M is a set, denote by $\mathfrak{M}$ the set of all subsets of M. A set M is said to have property (E), if there exists a single-valued mapping φ of $\mathfrak{M}$ into itself such that, for every subset A of M containing at least two elements, $\varphi(A)$ is a nonempty proper subset of A. It is shown that every set M having property (E) can be ordered. Call an ordered set M cleavably ordered, if there exists a single-valued mapping ψ of $\mathfrak{M}$ into itself such that, for every subset A of $\mathfrak{M}$ containing at least two elements, $\psi(A)$ is a nonempty proper initial segment of A. Each of the following conditions is necessary and sufficient for a set M to possess property (E): (1) there exists a cleavable ordering of M; (2) there exists a well-ordered set N, and a subset of $\mathfrak{N}$ that is equivalent to M. An ordered set M has property (E), if there exists a subset of M that is dense relative to M and can be well-ordered. *F. Bagemihl* (Notre Dame, Ind.).

Cuesta, N. Triadic construction of partially ordered sets. Acta Salmant. Ser. Ci. (N.S.) 1 (1955), no. 4, 42 pp.

Let M be a partially ordered set with the ordering relation $\prec$. A cavity of M [cf. Cuesta, Rev. Acad. Ci. Madrid 48 (1954), 103–145; MR 16, 1091] is a decomposition of M into three mutually exclusive sets I, N, F such that, if $x \epsilon I$, $y \epsilon F$, $m \epsilon M$, $m' \epsilon M$, $m \prec x$, and $y \prec m'$, then $x \prec y$, $m \epsilon I$, and $m' \epsilon F$. Relative to an arbitrarily chosen well-ordering of M: $m_1, m_2, \cdots, m_\xi, \cdots$ $(0<\xi<\mu, |\mu|=|M|)$, the cavity (I, N, F) is given the triadic name $0a_1a_2\cdots a_\xi\cdots$ $(\xi<\mu)$, where a_ξ is the letter i, n, f according as $a_\xi \epsilon I$, $a_\xi \epsilon N$, $a_\xi \epsilon F$. The given partial order can be extended by occupying a cavity of M with a new element which is made to assume the triadic name of this cavity. The triadic names of the cavities of the extension can be inferred from those of the cavities of the original partial order. This makes it possible to construct inductively (beginning with the null set) all the partial orderings of a given set; to each partial order there corresponds a genealogical tree whose elements are the tridiac names of the elements of the partially ordered set as well as of the cavities that appear in the successive stages of the construction. *F. Bagemihl* (Notre Dame, Ind.).

Kreĭnin, Ya. L. On sets effectively different from all Φ-sets. Mat. Sb. N.S. 38(80) (1956), 129–148. (Russian)

Let: R: metric space; $\pi(R)$: system of all ω-sequences of

closed sets; $F(R)$: metric space of all non-void closed sets for any bounded R; $\pi t(R)$: topological product of the ω-sequence of spaces $F(R)$ with the metric

$$\varrho((F_1, F_2, \cdots)(F_1', F_2', \cdots)) = \sum \frac{1}{n^2} \frac{\alpha(F_n, F_n')}{1+\alpha(F_n, F_n')}$$

($F_n, F_n' \in F(R)$). If ϕ is a δs-operation, then a set $M \subseteq R$ is called a ϕ-set, provided $M = \phi\{F_n\}$ with $\{F_n\} \in F(R)$. A ϕ-basis of R is any $\pi_\phi(R) \subseteq \pi(R)$ such that every ϕ-set M be a ϕ-set relative to $\pi_\phi(R)$. A set T_ϕ in R is said to be effectively distinct from all the ϕ-sets of R, if there exists a nonvoid bounded closed set Z_ϕ and a mapping ν_ϕ of a ϕ-basis $\pi_\phi(R)$ onto Z_ϕ such that:

a) $\nu_\phi\{F_n\} \in T_\phi \cap C\phi\{F_n\} \cup \phi\{F_n\} \cap CT_\phi$ ($\{F_n\} \in \pi_\phi(R)$);

b) $\pi t(Z_\phi) \subseteq \pi_\phi(R)$; c) ν_ϕ is continuous in $\pi t(Z_\phi)$. An effective λ-space is any R such that for each ϕ there exists a $C\phi$-set effectively different from all the ϕ-sets of R. R is a λ-space provided for each ϕ there is a $C\phi$-set in R which is not a ϕ-set. R is a γ-space if R is contained in a compactum and if there exists a continuous mapping of R onto the dyadic discontinuum (d.d.) D. A subset $\tilde{D}$ of R is a γ-nucleus of R if $\tilde{D}$ is a closed γ-set (i.e. $\tilde{D}$ as a relative space is a γ-space). The paper examines the conditions in order that R be a λ-space. According to Kolmogorov each R containing a D is a λ-space. Every γ-space is a λ-space (Th. 2.1). If R contains a γ-nucleus, then R is a λ-space (Th. 2.2) There exists an R containing no D and containing a γ-nucleus. The metric spaces $D^{\aleph_0}$, $F(D)$, $\pi t(D)$ are d.d. (Th. 5.1, 5.3, 5.4). § 4 contains some notions and results in connections with Novikov's effectiveness considerations [Izv. Akad. Nauk SSSR. Ser. Mat. **1939**, 35–40]. If R is an effective λ-space, then R contains D (Th. 5.3) and vice versa (Th. 6.1). *D. Kurepa.*

Šegol'kov, E. A. On a property of set-theoretical operations. Moskov. Gos. Ped. Inst. Uč. Zap. **71** (1953), 81–91. (Russian)

The following problem is studied: of a class of set-theoretic operations on sequences $E_1, E_2, \cdots$ of subsets of a set R, characterize the subclass of operations which are invariant under rearrangement of the sequence. Solutions are given for two classes of operations introduced by Kantorovich and Livenson [Fund. Math. **18** (1932), 214–279], of which one is the δs-operation. The solutions are too long to be given briefly. *E. E. Floyd.*

Froda, Alexandru. Ensembles des distances. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2**(1953), no. 3, 21–22. (Romanian. Russian and French summaries)

Let E be a point set of the euclidean n-space $\mathcal{E}^{(n)}$. Denote by $D(E)$ the set of distances of pairs of (distinct) points of E. If there exists a one-to-one correspondence between the set of pairs of points and $D(E)$, then E is said to be a set of unequal distances. If L is an arbitrary set of positive numbers, dense in $(0, +\infty)$, the problem discussed is that of constructing a set E, dense in $E^{(n)}$, and such that $D(E) \subset L$. The solution of the problem is known for $n=1$. The impossibility of constructing a set E dense in $\mathcal{E}^{(2)}$ and satisfying the conditions of the problem would imply that to any set E, dense in $\mathcal{E}^{(2)}$, corresponds a set $D(E)$ containing an infinity of irrational values. This last assertion is neither proved, nor disproved, but is said to be made plausible by the following result, previously proven by the author [Acad. R. P. Romîne. Stud. Cerc. Mat. **5** (1954), 29–71; MR **16**, 734]: Let $\mathcal{E} = \sum_{r=1}^{\infty} \mathcal{E}^{(r)}$ be the space of points of a finite (but unbounded) number of non-vanishing real cartesian coordinates. A set is called minimal, if each of its subsets contains the least number of points consistent with its dimension. Then there exist sets L, such that any point set E in $\mathcal{E}$, with unequal distances and such that $D(E) \subset L$, is minimal.

E. Grosswald (Philadelphia, Pa.).

Császár, Á. Sur la structure des ensembles de niveau des fonctions réelles à une variable. Colloq. Math. **4** (1956), 13–29.

Let $\mathfrak{R}$ denote any family of sets of real numbers x having the following sole properties: (1) $B \subset A$, $A \in \mathfrak{R}$, implies $B \in \mathfrak{R}$; (2) $A_i \in \mathfrak{R}$, $i=1, 2, \cdots$, implies $\sum A_i \in \mathfrak{R}$. Let $\overline{\mathfrak{R}}$ be the family of all sets of real numbers x of the form $N+H$ where $N \in \mathfrak{R}$ and H is countable. Let $f(x)$, $-\infty < x < +\infty$, be a real single-valued function and, for any x_0 and $d>0$, let E, E^* be the sets $E_x[f(x) \leqq f(x_0)$, $x_0 < x < x_0+d]$, $E_x[f(x) < f(x_0)$, $x_0 < x < x_0+d]$. Then $f(x)$ is said to be increasing at the right of x_0 with respect to $\mathfrak{R}$ or is $C_+(\mathfrak{R})$ at x_0 if $E \in \mathfrak{R}$ for some $d>0$. Analogously $f(x)$ is said to be non-decreasing [oscillant] at the right of x_0 with respect to $\mathfrak{R}$ or is $C_+^*(\mathfrak{R})$ at x_0 [$O_+(\mathfrak{R})$ at x_0] if $E^* \in \mathfrak{R}$ for some $d>0$ and $E \notin \mathfrak{R}$ for all $d>0$ [$E \notin \mathfrak{R}$, $E^* \notin \mathfrak{R}$ for all $d>0$]. Also, analogous concepts D_+, D_+^*, C_-, C_-^*, D_-, D_-^*, O_- are defined. Properties C, D, O are used to characterize certain situations which are in a certain sense exceptional. Here are some of the theorems. I. If $f(x)$ is $C_+^*(\mathfrak{R})$ [or $C_-^*(\mathfrak{R})$, or $D_+^*(\mathfrak{R})$, or $D_-^*(\mathfrak{R})$] at every point of a set E, then $E = N+Q$ where $N \in \mathfrak{R}$ and $f(Q)$ is countable. II. If $f(x)$ is $C_+^*(\mathfrak{R})$, $D_-^*(\mathfrak{R})$ at every point of a set E, then $E = N+R$ where $N \in \overline{\mathfrak{R}}$ and R can be covered by a countable family of open intervals on which $f(x)$ is constant but for a subset of points which belongs to N. For other statements we refer to the paper. [For extensions to functions of two real variables see Á. Császár, Acta. Sci. Math. Szeged **15** (1954), 183–202; MR **16**, 343.] *L. Cesari* (Lafayette, Ind.).

Fodor, G. Some results concerning a problem in set theory. Acta Sci. Math. Szeged **16** (1955), 232–240.

Let $|S| = \mathfrak{m} \geqq \aleph_0$, $Q \subseteq S$, $|Q| = \mathfrak{q} \geqq \aleph_0$, and $\mathfrak{p} \leqq \mathfrak{m}$. Suppose that, for some $\mathfrak{n} < \mathfrak{m}$, to every $x \in Q$ there corresponds a set $H_x \subset S$ with $|H_x| < \mathfrak{n}$, such that $|\mathbf{U}_{x \in Q} H_x| = \mathfrak{q}$. A subset Γ of Q is said to have property $T(\mathfrak{q}, \mathfrak{p})$ provided that $|\mathbf{U}_{x \in \Gamma} H_x| = \mathfrak{q}$ and $|\mathbf{U}_{x,y \in \Gamma; x \neq y} (H_x \cap H_y)| < \mathfrak{p}$. The problem considered is: if $\mathfrak{n} < \mathfrak{q}$ and $\mathfrak{n} \leqq \mathfrak{p} \leqq \mathfrak{q}$, must there exist a subset Γ of Q having property $T(\mathfrak{q}, \mathfrak{p})$? (If the answer to this problem for $\mathfrak{p} = \mathfrak{q}$ is affirmative, then the answer to the problem of Ruziewicz in the theory of relations [cf. Erdös, Proc. Amer. Math. Soc. **1** (1950), 127–141; MR **12**, 14] is also affirmative when the power of the given set is greater then $\aleph_1$ and the given bounding cardinal number is infinite). Some conditions on $\mathfrak{n}$, $\mathfrak{p}$, $\mathfrak{q}$ are obtained under which the answer to the problem is affirmative (e.g., $\mathfrak{n} = \mathfrak{p} = \aleph_0$ and $\mathfrak{q}$ is regular), and others (credited to Erdös) under which the answer is negative (e.g., $\mathfrak{q}$ is singular and $\mathfrak{p} \leqq \mathfrak{q}'$, where $\mathfrak{q}'$ is the immediate successor of the smallest cardinal number $\mathfrak{k}$ such that $\mathfrak{q}$ is the sum of $\mathfrak{k}$ cardinal numbers each of which is less than $\mathfrak{q}$). *F. Bagemihl.*

Schmidt, Jürgen. Konfinalität. Z. Math. Logik Grundlagen Math. **1** (1955), 271–303.

The author discusses the cofinal types of directed systems after the pattern first introduced by Tukey [Convergence

and uniformity in topology, Princeton, 1940; MR 2, 67]. After discussing the partial ordering of the family of cofinal types and the minimality of the cofinal types of well-ordered systems, he introduces some cardinal and ordinal number invariants of cofinal types and is able, by means of these, to find the relations between the cofinal types of most of the systems $\tau_{\alpha\beta}$, where $\beta \leqq \alpha$ are ordinal numbers and $\tau_{\alpha\beta}$ is the set of all subsets of cardinal number $<\aleph_\beta$ of a set E of cardinal number $\aleph_\alpha$. *M. M. Day.*

Padmavally, K. A remark on order-types. Fund. Math. 42 (1955), 312–318.

Define $\exp a = 2^a$, $\aleph_{\sigma(k)} = \exp \aleph_k$, and for every continuous order type C, define $l(C)$ = least l such that C contains neither ω_l nor ω_l^*, $\aleph_{m(C)}$ = least cardinal of the dense subsets of C, and $n(C)$ = least n such that C is embeddable in the lexicographically ordered set of all dyadic ω_n-sequences (reviewer's notation). J. Novák [Fund. Math. 39 (1952), 53–64; MR 15, 17] proved that $n(C) \leqq m(C)$, and the author [Rev. Mat. Hisp.-Amer. (4) 14 (1954), 50–73; MR 16, 119] proved that $n(C) \leqq l(C)$. The main results in the present paper are as follows. Theorem 2.1: $n(C) = m(C)$ if and only if $\sum_{x<\alpha} \aleph_x = \aleph_\alpha$. Theorem 2.2: The relation $m(C) \leqq l(C)$ for all C is equivalent to the generalized continuum hypothesis. (This is related to a theorem of the reviewer's [Ann. of Math. (2) 56 (1952), 440–459, Th. 3.3 ff.; MR 14, 543].) Theorem 3.5: For every ordinal k, there exists an increasing sequence of power $\aleph_{\sigma(k)+1}$ of types C_β, each of power $\leqq \aleph_{k+1}$, and with each $l(C_\beta) \leqq k+1$. This generalizes a result of Sierpiński [Fund. Math. 37 (1950), 253–264; MR 13, 19]. *L. Gillman.*

Neumer, Walter. Über Folgen von Ordnungszahlen. Z. Math. Logik Grundlagen Math. 1, 109–126 (1955).

Let M and N be subsets of $W(\omega_\alpha)$ such that $\overline{\overline{M}} = \overline{\overline{N}} = \omega_\alpha$. Define $M \succ N$ to mean that $\overline{\overline{N-M}} < \aleph_\alpha$ and $\overline{\overline{M-N}} = \aleph_\alpha$. (If ω_α is regular, then $M \succ N$ means that M contains a remainder of N but N contains no remainder of M.) Define $M \,{}_*\!\succ N$ to mean that M contains a remainder of N and $\overline{M-N} = \omega_\alpha$. An ω_α-part is a subset, of type ω_α, of $W(\omega_\alpha)$; an ω_α-band is a closed ω_α-part. A $\succ$-chain, resp. $\succ$-sequence, is an ordered, resp. well-ordered, subset of the set of all ω_α-parts partially ordered by $\succ$, and an ${}_*\!\succ$-sequence is a subset, well-ordered by ${}_*\!\succ$, of the set of all ω_α-parts. Say that a $\succ$-sequence (${}_*\!\succ$-sequence) S_1 can be prolonged, if these exists a $\succ$-sequence (${}_*\!\succ$-sequence) S_2 of which S_1 is an initial segment. Among the results that the author obtains concerning $\succ$-sequences and ${}_*\!\succ$-sequences are the following. Every $\succ$-sequence of type ω_α and every ${}_*\!\succ$-sequence of type ω_α can be prolonged. If ω_α is regular and $\lambda < \omega_{\alpha+1}$ is a limit number that is not cofinal with ω_α, then there exists a $\succ$-sequence of type λ which cannot be prolonged. If $\omega_{\alpha+1} \leqq \beta < \omega_{\alpha+2}$, there exists a $\succ$-sequence, of type β, of ω_α-parts M_ν ($\nu < \beta$), where M_0 can be chosen arbitrarily; this holds if "$\succ$-sequence" is replaced by "${}_*\!\succ$-sequence". If ω_α is regular and $\omega_{\alpha+1} \leqq \beta < \omega_{\alpha+2}$, then there exists a $\succ$-sequence, of type β, of ω_α-bands B_ν ($\nu < \beta$), where the ω_α-band B_0 can be chosen arbitrarily. If ω_α is regular, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ implies that there exists a $\succ$-sequence, of type $\omega_{\alpha+1}$, of ω_α-bands, which cannot be prolonged by the adjunction of an ω_α-part [this generalizes a result of Sierpiński, Czechoslovak Math. J. 1(76) (1951), 97-101; MR 13, 828]. There are further results for singular ω_α, and the relation between $\succ$ and ${}_*\!\succ$, and the behavior of ω_α-bands, is studied for this case. The insertion of ω_α-parts in $\succ$-chains is considered for regular ω_α.

F. Bagemihl (Notre Dame, Ind.).

Poprużenko, J. Sur certains ensembles indénombrables singuliers de nombres irrationnels. Fund. Math. 42 (1955), 319–338.

Let M be a nonenumerable set partially ordered with respect to a relation $\prec$, and suppose that, if $\alpha \leqq \omega$ and $m_0 \prec m_1 \prec \cdots \prec m_\nu \prec \cdots$ ($m_\nu \in M$, $\nu < \alpha$), then there exists an element $g \in M$ with $m_\nu \prec g$ for every $\nu < \alpha$. A subset N of M is said to be fundamental (with respect to $\prec$), provided that the subsets of N of power less than $|N|$ are characterized by the property of being bounded (a subset T of M is bounded, if there exists an element $r \in M$ such that $t \prec r$ for every $t \in T$). It is shown that there exist nonenumerable fundamental subsets of M. The author is interested in the case where M is the set of irrational numbers in the unit interval and, if b and c are elements of M whose expansions into infinite simple continued fractions are

$$b = \frac{1}{|b_1|} + \frac{1}{|b_2|} + \cdots + \frac{1}{|b_n|} + \cdots,$$

$$c = \frac{1}{|c_1|} + \frac{1}{|c_2|} + \cdots + \frac{1}{|c_n|} + \cdots,$$

we write $b \prec c$ provided that for some natural number k we have $b_n < c_n$ for $n \geqq k$. Let E be a fundamental subset of this set of irrational numbers. Then $|E| > \aleph_0$, and there exists a sequence $\{p_n(x)\}$ of polynomials, convergent on the closed unit interval, such that (1) no subsequence of $\{p_n(x)\}$ converges uniformly on any subset of E of power $|E|$, and (2) if $E_1 \subset E$ and $|E_1| < |E|$, there exists a sequence $\{F_k\}$ of closed sets whose union contains E_1, such that $\{p_n(x)\}$ converges uniformly on every F_k ($k = 1, 2, 3, \cdots$). A set X of irrational numbers in the unit interval is fundamental if, and only if, $\{p_n(x)\}$ is not σ-uniformly convergent [cf. Popruzenko, Fund. Math. 41 (1954), 29–37; MR 16, 343] on any subset of X of power $|X|$, but is σ-uniformly convergent on every subset of X of power less than $|X|$. This implies that the relation $2^{\aleph_0} = \aleph_1$ is true if, and only if, there exists a nonenumerable subset of the unit interval possessing, property (L) [cf. Sierpiński, Hypothèse du continu, Warszawa-Lwów, 1934, p. 37] and every fundamental set of irrational numbers is of the power of the continuum. If S is a fundamental set of irrational numbers, then there exists an infinite set R_S of rational numbers such that, for every open set G containing R_S, $|S-G| < |S|$.

F. Bagemihl (Notre Dame, Ind.).

Davies, Roy O. A note on linear derivates of measurable functions. Proc. Cambridge Philos. Soc. 52 (1956), 153–155.

The ray issuing from a point in the direction θ is denoted by $P(\theta)$. Assuming the axiom of choice, the author proves the existence of a plane set E of measure zero with the property that at almost all points P of the plane, the set of directions θ such that P is an accumulation point of $EP(\theta)$ is of full outer measure, and so also is the set of directions θ such that $EP(\theta)$ is empty. Hereby the measure of a set S of directions in the plane means the linear measure of the set of points Q on a unit circle for which the direction from the center to Q belongs to S. A corollary states that the upper linear derivate in the direction θ of a measurable function $f(x, y)$ is not necessarily a measurable function of the three variables x, y, θ. *A. Rosenthal.*

Piranian, George. The derived sets of a linear set. Michigan Math. J. 3 (1955–56), 83–84.

For any set E on the x-axis R there is defined an ordinal-valued function $p(x, E)$ over R, where $p(x, E)$ is the order of the maximal derived set of E (of order $\leq\Omega$) containing x. The author gives necessary and sufficient conditions that an arbitrary ordinal-valued function $p(x)$ over R be a function $p(x, E)$ for some $E \subset R$. The conditions are clearly all necessary and independent. The argument for sufficiency is straightforward. *R. D. Anderson.*

Lipiński, J. S. Une propriété des ensembles $\{f'(x)>a\}$. Fund. Math. 42 (1955), 339–342.

It was proved by Z. Zahorski [Trans. Amer. Math. Soc. 69 (1950), 1–54; MR 12, 247] that, if a function $\phi(x)$ of one real variable has a derivative (finite or infinite) at all points, then the set $\{\phi'(x)>a\}$ belongs to a certain class M_2; and if moreover $\phi'(x)$ is everywhere finite then the set $\{\phi'(x)>a\}$ belongs to a more restricted class M_3. He asked the question whether, for every set E of M_2, there is a function $\phi(x)$ such that $\{\phi'(x)>0\}$ is identical with E, and also a similar question concerning M_3. In this paper the author answers both these questions in the negative. He shows first that if $E=\{\phi'(x)>a\}$ then a certain related set A_E is non-dense, and then constructs a set of class M_3 (and consequently also of class M_2) which has not this further necessary property. *U. S. Haslam-Jones.*

Marcus, S. Les ensembles F_σ et la continuité symétrique. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 871–886. (Romanian. Russian and French summaries)

A real function of a real variable is called symmetrically continuous at x if and only if $\lim_{h\to 0}(f(x+h)-f(x-h))=0$. The author proves that if D is a real F_σ set, then there is a function f such that D is precisely the set of points where f is not symmetrically continuous. *M. M. Day.*

Fraile, Vicente. Finite algorithms for periodic graphs. An. Asoc. Españ. Progr. Ci. 20 (1955), 7–11. (Spanish)

A single formula for step-functions with an application to the approximation of Riemann-integrable functions by step-functions. *T. A. Botts* (Charlottesville, Va.).

McKiernan, Michel. On the nth derivative of composite functions. Amer. Math. Monthly 63 (1956), 331–333.

Froda, Alexandre. Propriétés (à distance) des fonctions réelles dans un espace euclidien. C. R. Acad. Sci. Paris 242 (1956), 1948–1951.

The author considers real functions defined on Euclidean space E^n, $n>1$. If $\varrho>0$ is a given constant, suppose there is attached to each point $P \in E^n$ a variety $J(P)$, starlike from P, at a distance from P which is $\geq\varrho$ and $\leq 2\varrho$. Then, instead of considering the values of a real function f near P, he considers the values of f on $J(P)$ and obtains theorems without making assumptions about the class of f. For example, certain sets of constancy can be described. *M. E. Shanks* (Los Angeles, Calif.).

Gegeliya, T. G. On a generalization of a theorem of G. Giraud. Soobšč. Akad. Nauk Gruzin 16 (1955), 657–663. (Russian)

Let S be a surface sufficiently regular (we omit the specific definition), $r(P, Q)$ the distance between the points P and Q, and $\varphi(Q)$ a continuous function defined on S and such that the integral

$$\iint_S \frac{|\varphi(Q)-\varphi(P)|}{r^2(P, Q)}\,dS_Q$$

converges uniformly on S. By $\omega(\delta, \varphi)$ we denote the modulus of continuity of φ. The main purpose of the note is to estimate the modulus of continuity of the function

$$\chi(P)=\iint_S \frac{M(Q, P)[\varphi(Q)-\varphi(P)]}{r^2(Q, P)}\,dS_Q,$$

where $M(Q, P)$ is a given function, bounded and satisfying, for example, the condition $|M(Q, P_1)-M(Q, P_2)|\leq C_r(P_1, P_2)$ (the author considers a less restrictive condition). The main result of the note is that then

$$\omega(\delta, \chi)\leq C[\omega(\delta, \varphi)+\int_0^\delta \tau^{-1}\omega(\tau, \varphi)d\tau+\delta\int_\delta^\eta \tau^{-2}\omega(\tau, \varphi)d\tau],$$

where C and η are positive constants. The case when $\varphi \in \mathrm{Lip}\,\alpha$, $0<\alpha<1$, was considered previously by Giraud [Ann. Sci. Ecole Norm. Sup. (3) 51 (1934), 251–372]. *A. Zygmund* (Chicago, Ill.).

Albrycht, J. Some remarks on the Marcinkiewicz-Orlicz space. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 1–3.

Let M and N be complementary functions in the sense of Birnbaum and Orlicz [Studia Math. 3 (1931), 1–67]. The author adapts Orlicz's definition [Bull. Internat. Acad. Polon. Sci. Lett. Cl. Sci. Math. Nat. Sér. A. 1932, 207–220] from finite to infinite intervals by averaging over increasingly long finite intervals (as Marcinkiewicz [C. R. Acad. Sci. Paris 208 (1939), 157–159] approached the almost periodic functions of Besicovitch). This defines a pseudo-normed space; the quotient space by the space of elements with pseudo-norm zero is asserted to be a complete, non-separable normed space. *M. M. Day.*

See also: Kondô, p. 933; Zeeman, p. 939; Eggleston, p. 957.

Theory of Measure and Integration

Onicescu, O. Intégrales singulières et représentation des fonctionelles linéaires. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 1 (1953), no. 2, 15–18. (Romanian. Russian and French summaries)

The author proves that if f is continuous on $[a, b]$ and if F is a monotone function defined on the whole line with total increase one, then

$$\lim_{\alpha\to 0}\int_a^b f(x)dF\left(\frac{x-\xi}{\alpha}\right)=\begin{cases}0 \text{ if } \xi<a \text{ or } \xi>b,\\ f(\xi) \text{ if } a<\xi<b.\end{cases}$$

The result is extended to functions of bounded variation and to n dimensions. Combined with Helly's theorem on limits of functions of bounded variation this leads to a proof of the F. Riesz theorem on representation of a linear functional on the space of continuous functions on $[a, b]$. *M. M. Day* (Urbana, Ill.).

Onicescu, O.; et Ionescu Tulcea, C. T. Sur la régularisation des mesures et le théorème de Riesz. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 4 (1955), no. 6–7, 35–38. (Romanian. Russian and French summaries)

Given an interval I in Euclidean n-space, the author

constructs on the product $I\times I$ a sequence of density functions whose total weight is concentrated nearer the diagonal with increasing n. With these functions he gives short proofs that (1) each additive bounded variation interval function on I is the "weak" limit of such functions defined from continuous density functions, and (2) each linear functional on $C(I)$ is representable as an integral with respect to such a function (Riesz theorem).

M. M. Day (Urbana, Ill.).

Caccioppoli, Renato. L'integrazione e la ricerca delle primitive rispetto ad una funzione continua qualunque. Ann. Mat. Pura Appl. (4) **40** (1955), 15–34.

For each continuous function $\varphi(x)$, $a\leqq x\leqq b$, the author introduces certain "generating sequences" $\{\varphi^{(n)}(x)\}$ such that each $\varphi^{(n)}(x)$ is of bounded variation and $\lim \varphi^{(n)}(x)=\varphi(x)$ uniformly in x. A set E is said to be of φ-measure zero if E has measure zero with respect to each measure $d\varphi^{(n)}(x)$. If E has φ-measure zero, then it satisfies the condition

$$(*)\qquad \liminf_{\delta\to 0}\sum_i |\varphi(x_i'')-\varphi(x_i')|=0,$$

where $\{(x_i''-x_i')\}$ is a sequence of non-overlapping intervals covering E and $\delta=\sup(x_i''-x_i')$, but not conversely. $F(x)$ is said to be absolutely continuous with respect to $\varphi(x)$ if (*) implies $\lim \sum_i |F(x_i'')-F(x_i')|=0$. A function $f(x)$ is said to be φ-integrable if all the functions $F^{(n)}(x)=\int_a^x f(t)d\varphi^{(n)}(t)$ are equally-absolutely continuous with respect to the functions $\varphi^{(n)}(x)$. In this case $F^{(n)}(x)$ converges to a limit function $F(x)$ which is denoted by $F(x)=\int_a^x f(t)d\varphi(t)$, and the following theorems are true: (1) F is φ-absolutely continuous if and only if the derivative $f=dF/d\varphi$ exists with exeption of a set of φ-measure zero and $F(x)-F(a)=\int_a^x f(t)d\varphi(t)$. (2) If F is φ-absolutely continuous and $dF/d\varphi=0$ with exception of a set E satisfying (*), then $F=$constant. (3) The Vitali theorem on term-by-term integration holds for φ-integrable functions. (4) The notion of φ-integrability is independent of the choice of the generating sequence. Finally, the author outlines briefly the theory of Denjoy totalization with respect to $\varphi(x)$, obtaining thus (for the real case) results similar to those of G. Choquet [J. Math. Pure Appl. (9) **26** (1947), 115–226; MR **9**, 419]. *M. Cotlar.*

Džvaršeišvili, A. G. On generalized absolutely continuous functions of two variables. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 77–110. (Russian)

The classical theory of Lebesgue and the theory of Looman-Kryżański of n-dimensional Denjoy integrals have been generalized by Banach, Saks, Burkill and Kempisty [Kempisty, Fund. Math. **27** (1936), 10–37] for non-additive set-functions. However, these authors deal only with ordinary derivatives and restricted $\mathfrak{D}$-integrals. The theory of the double $\mathfrak{D}$-integral, with approximate derivatives, has been developed by Čelidze [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **15** (1947), 155–242; MR **14**, 735], who also introduced a new definition of absolute continuity. In the present paper the author extends the theory of Čelidze for non-additive interval-functions $f(I)$, in the fashion Kempisty extended that of Looman-Kryżański. For instance, he introduces the Burkill integral $B_E(f, R)$ of $f(I)$ with respect to the set E, and proves that if $f\in$ (AC) in the sense of Čelidze, and if $D_{ap}f(p)\geqq 0$ almost everywhere, then $B_E(f, R)\geqq 0$, and conversely. Using these results the author derives some new properties of the functions (ACG), for instance: (1) Let

$$F(x,y)=\int_a^x dt\int_c^y f(t,r)dr,$$

$$G(x,y)=\int_c^y dr\int_a^x f(t,r)dt \text{ (Denjoy integrals)},$$

then $F=G$ if and only if F and G are (ACG) in the sense of Čelidze. (2) If $F(x, y)\in$ (ACG), then each perfect set E contains a portion A such that

$$F(r)=(L)\int_r\int_E D_{ap}F(x,y)dx\,dy+B_A(g,r),$$

where $g(r)=0$ if two opposite end-points of the interval r belong to A, and $g(r)=F(r)$ otherwise. (3) If $F\in$ (ACG), then $D_{ap}F(x,y)=\partial_{ap}{}^2F(x,y)/\partial x\partial y=\partial_{ap}{}^2F/\partial y\,\partial x$, almost everywhere. *M. Cotlar* (Mendoza).

Viola, Tullio. Sulle successioni di funzioni quasi continue negli spazi astratti. Ann. Mat. Pura Appl. (4) **39** (1955), 381–391.

S: espace métrique séparable, S': groupe abélien normé complet, S'': espace métrique, compact et séparable. Mesure μ: fonction définie sur la famille $\mathfrak{F}$ des ensembles boréliens de S, prenant ses valeurs dans S', complètement (dénombrablement) additive et de variation totale m (définie sur $\mathfrak{F}$) finie. Un ensemble $I\subset S$ est dit "lebesguien" si à tout $\varepsilon>0$ correspondent un ensemble ouvert $A_\varepsilon\supset I$ et un ensemble fermé $C_\varepsilon\subset I$ tels que $m(A_\varepsilon-C_\varepsilon)<\varepsilon$. Une fonction f définie sur un ensemble $I\subset S$ et prenant ses valeurs dans S'' est dite "quasi-continue" si I est lebesguien et si à tout $\varepsilon>0$ correspond un ensemble fermé $C_\varepsilon\subset I$ tel que $m(I-C_\varepsilon)<\varepsilon$ et que f soit continue sur C_ε. Théorème: Si f_n est une suite de fonctions définies et quasi-continues sur un même ensemble $I\subset S$ et prenant leurs valeurs dans S'', il existe une fonction quasi-continue sur I égale en presque tout point P (relativement à m) de I à une des valeurs limites de la suite des $f_n(P)$. {Remarques du rapporteur: La mesure μ n'intervient que pour définir m qui est une mesure finie non négative définie sur $\mathfrak{F}$, par ailleurs quelconque. La topologie de S joue aussi un rôle accessoire. En effet, un ensemble lebesguien n'est autre qu'un ensemble mesurable relativement à l'extension complète m^* de m, une fonction quasi-continue est une fonction m^*-mesurable, entendant par là que, pour tout ensemble fermé c'' de S'', $f^{-2}(C'')$ est m^*-mesurable. Un espace métrique compact est toujours séparable.} *C. Y. Pauc* (Nantes).

Davies, Roy O. Non σ-finite closed subsets of analytic sets. Proc. Cambridge Philos. Soc. **52** (1956), 174–177.

Every analytic set (Suslin set), in a Euclidean space, which is non-σ-finite with respect to Hausdorff Λ^α-measure contains a closed subset which is also non-σ-finite. *H. D. Ursell* (Leeds).

Fell, J. M. G. The measure ring for a cube of arbitrary dimension. Pacific J. Math. **5** (1955), 513–517.

This paper brings complements to D. Maharam's theorem on the structure of homogeneous measure algebras [Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 108–111; MR **4**, 12]. A non-zero element x of a measure ring R is called "homogeneous of order α" if the principal ideal R_x (which is a measure algebra) is so. The "cardinal function" ϕ of R associates with each cardinal α the smallest cardinal number $\phi(\alpha)$ of a maximal disjoint family of elements of

R which are homogeneous of order α. Theorem: The measure ring R is determined to within isomorphisms by its cardinal function ϕ. Two finite measures μ and ν defined on an additive class of sets F are said to be of the same Hellinger type if they have the same null sets. For a fixed F the Hellinger types partially ordered with respect to the relation of absolute continuity form a measure ring. Theorem: If α is an infinite cardinal, the cardinal function characterizing the measure ring $R^{(\alpha)}$ of Hellinger types of measures on the Baire subsets of the α-dimensional cube is given by $\phi(\gamma)=\gamma^{\alpha}$ if γ is a 0 or infinite cardinal number with $\gamma\leqq\alpha$, $=0$ otherwise. *C. Y. Pauc.*

Phakadze, Š. S. Extendability of solvable classes. Soobšč. Akad. Nauk Gruzin SSR **16** (1955), 761–768. (Russian)

As well as can be told from the usage, a solvable class M of sets in Euclidean n-space R^n is a Borel field of sets on which a non-trivial countably additive measure function can be defined. It is shown here that no such family can be maximal; that is, that if such a solvable family M with measure μ is given there always exists a family M' properly containing M and a measure μ', which is an extension of μ, defined on M'. The proof depends on the result of S. Ulam [Fund. Math. **16** (1930), 140–150] that there is no non-trivial 2-valued measure defined on the set of all subsets of a set of the cardinal of the continuum. *M. M. Day* (Urbana, Ill.).

Phakadze, Š. S. Nonmeasurable sets absolutely of zero measure, their denumerable sums and properly almost invariant sets. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 343–350. (Russian)

A set M' in Euclidean n-space R^n is called a configuration of M if M' is contained in the union of countably many congruent copies of M; A is absolutely of measure zero in the strong sense if each configuration of A has a complement so large that two congruent copies of it cover R^n. The author states several results using these concepts and some generalizations of them to obtain results related to those of Sierpiński [Comment. Math. Helv. **22** (1949), 317–320; MR **10**, 684]. For example, if F is a family of one-to-one transformations of R^n onto itself of cardinal less than the continuum, then R^n can be decomposed into continuum-many disjoint, almost F-invariant sets, i.e., sets E of positive Lebesgue outer measure, even after removal of any set of cardinal less than that of the continuum, such that the symmetric difference of E and $f(E)$ is of cardinal less than the continuum for every $f\in F$. No proofs are given. *M. M. Day* (Urbana, Ill.).

See also: Davies, p. 952; Potapov, p. 958.

Functions of a Complex Variable, Generalizations

Briggs, William E. Some constants associated with the Riemann zeta-function. Michigan Math. J. **3** (1955–56), 117–121.

The constants are the γ_n in the expansion

$$\zeta(s)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n}{n!}(s-1)^n.$$

It is shown that infinitely many of the γ_n satisfy each of the inequalities $\gamma_n>0$, $\gamma_n<0$, $n^{-\varepsilon n}<|\gamma_n|<n^{\varepsilon n}$, where $\varepsilon>0$ is arbitrary. It is also shown that $|\gamma_n|<(n/2e)^n$ for all γ_n. *H. S. Zuckerman* (Seattle, Wash.).

Stoka, Marius I. Sur une interprétation géométrique des transformations linéaires du plan complexe et des similitudes du plan réel. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 65–75. (Romanian. Russian and French summaries)

Let $(y_0+y_1)(\xi^2+\eta^2)-2y_2\xi-2y_3\eta+y_0-y_1=0$ be the equation of a circle K in the (ξ,η)-plane. K is uniquely determined by the y_i's ($i=0, 1, 2, 3$), which enter the equation homogeneously and are called tetracyclic coordinates of K. One may also interpret the y_i's as the homogeneous coordinates of a point Y in 3-dimensional euclidean space $E^{(3)}$. Thus a 1-1 correspondence has been established between the circles K of (ξ,η) and the points of $E^{(3)}$. The same correspondence is obtained by stereographic projection from the south pole S of the unit sphere, as previously shown by the author [Gaz. Mat. Fiz. **5** (1953), 241–253]. If one considers the complex variable $z=\xi+i\eta$ and performs upon it the linear transformation $z^*=\lambda z+\mu$, then the circle K is transformed into a circle K^*, of tetracyclic coordinates $\{y_i^*\}$ ($i=0, 1, 2, 3$), projectively related to the $\{y_i\}$ of K. If Y_j ($j=1, 2, 3, 4$) are the double points of that projectivity, then $Y_1=S$, Y_2 is on the unit sphere, while Y_3 and Y_4 are complex conjugate points in the plane tangent to the unit sphere at S, unless $\mu=0$. If M is the (real) midpoint of Y_3 and Y_4, it is shown that Y_2 and M completely determine the projectivity. If $Y_2=S$, $\lambda=1$ and conversely. Several further results are proven, of which the following is characteristic: If the straight line Y_2Y_3 remains pointwise invariant, $|\lambda|=1$, $\lambda\neq\pm1$. *E. Grosswald.*

Kaz'min, Yu. A. Infinite systems of linear equations and bases of analytical functions. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 179–182. (Russian)

The following two general theorems on bases for analytic functions are stated, and numerous particular consequences derived: 1. The system of functions (1) $f_k(z)=Z^k+\sum_{n=0}^{\infty}a_{nk}z^n$ ($k=0, 1, 2, \cdots$) is a "quasi-power" basis [for definition see Haplanov, same Dokl. (N.S.) **80** (1951), 177–180; MR **13**, 357] in the space of functions analytic in $|z|<r$, $1<r\leqq R$ if (2) $\sum_{n,k=0}^{\infty}|a_{nk}|<\infty$; (3) $\limsup_{n\to\infty}q_n^{1/n}=1/R$ where $q_n=\sup_{k\neq n}|a_{nk}|$;

$$(4)\qquad \Delta=\begin{vmatrix}(1+a_{00}) & a_{01} & \cdots\\ a_{10} & (1+a_{11}) & \cdots\\ \cdot & \cdot & \cdots\end{vmatrix}\neq 0.$$

2. System (5) $f_k(z)=z^k+\sum_{n=-\infty}^{+\infty}a_{nk}z^n$ ($k=0, k\pm1, \pm2, \cdots$) is a quasi-power basis for the ring $1/r_1<|z|<r_2$ with $1<r_1\leqq R_1$, $1<r_2\leqq R_2$ if (6) $\sum_{n,k=-\infty}^{+\infty}|a_{nk}|<\infty$;

$$(7)\qquad \limsup_{n\to\infty}q_{-n}^{1/n}=R_1^{-1},\quad \limsup_{n\to\infty}q_n^{1/n}=R_2^{-1},$$

where $q_n=\sup_{k\neq n}|a_{nk}|$ ($n=0, \pm1, \cdots$); (8) $\Delta\neq0$, where this Δ is similar to (4) but goes to infinity in all four directions.

We cite only two of the particular corollaries: (a) Let $f(z)=\sum_0^{\infty}a_nz^n/n!$ with $\sum_0^{\infty}|a_n|<1$. Then $f_k(z)=z^k+f^{(k)}(z)$ ($k=0, 1, \cdots$) is a quasi-power basis for $|z|<r$ for arbitrary $r>1$. (b) Let $\{\varphi_k\}$ ($k=0, \pm1, \cdots$) be such that $\limsup_{k\to\infty}|\varphi_{-k}|^{1/k}=R_1^{-1}<1$, $\limsup_{k\to\infty}|\varphi_k|^{1/k}=R_2^{-1}<1$, $\sup|\varphi_k|\leqq R_1^{-1}$. Then the systems $\{z^ke^{\varphi_k z}\}$, $\{z^k\cos\varphi_k z\}$ ($k=0, \pm1, \cdots$) form a quasi-power basis in any ring $1/r_1<|z|<r_2$ for which $1<r_1\leqq R_1$, $1<r_2\leqq R_2$. Remark: The article itself nowhere involves the "infinite systems of equations" mentioned in the title. The presumption is that such systems arise in the proofs (which are not given) of the theorems. *I. M. Sheffer* (University Park, Pa.).

Gabib-zade, A. Š. On a boundary problem of analytic functions. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **3** (1948), 85–90. (Russian. Azerbaijanian summary)

Let G be a bounded, simply connected region in the plane of the complex variable z, bounded by a curve L possessing continuous curvature. The following boundary problem is considered. It is required to find n functions $\varphi_1(z), \varphi_2(z), \cdots, \varphi_n(z)$, holomorphic in G, subject to the conditions that they, as well as all their derivatives of all orders $k \leqq N$, shall be continuous in the closure of G and, on L, shall satisfy Hölder conditions and the relations

$$(1)\quad \tfrac{1}{2}\mathrm{Re}\sum_{j=0}^{n}\sum_{k=0}^{N} a_{\lambda_{jk}}(t)\varphi_j^{(k)}(t)=b_\lambda(t)\ (\lambda=1, 2, \cdots, n),$$

where the $a_{\lambda_{jk}}(t)$, $b_\lambda(t)$ are prescribed functions on L satisfying Hölder conditions. The system of equations (1) is reduced to a system of integrodifferential equations and the relations of this system to the system (1) are discussed. *W. Seidel* (Notre Dame, Ind.).

Jenkins, James A. Some theorems on boundary distortion. Trans. Amer. Math. Soc. **81** (1956), 477–500.

Let $w=f(z)$ be univalent in a domain D, satisfy $|f(z)|<R$ there, and admit a continuous boundary correspondence between an arc A on the boundary of D and an arc Γ of $|w|=R$. Under these and various additional hypotheses the author constructs the functions which make the length of Γ maximal or minimal. *W. K. Hayman.*

Suvorov, G. D. On the order of equicontinuity of a class of univalent mappings in closed domains. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 22–23. (Russian)

Let $w=T(z)=f_1(x, y)+if_2(x, y)$, $T(0)=0$, map the bounded, simply connected region D, containing $\{|z|<\delta\leqq 1\}$, topologically onto a region Δ. The f_i are assumed to satisfy certain regularity conditions; in particular,

$$I_D(T)=\int_\Delta\int\sum_{i=1}^{2}\left[\left(\frac{\partial f_i}{\partial x}\right)^2+\left(\frac{\partial f_i}{\partial y}\right)^2\right]dx\,dy\leqq k<\infty.$$

[Cf. Lelong-Ferrand, J. Math. Pures Appl. (9) **31** (1952), 103–126, 245–252; MR **14**, 36.] Then if z_1 and z_2 are interior or "boundary" points of D, and $0<\varrho(z_1, z_2; D)<3^{\frac{1}{2}}\delta 2^{-\delta-2}$ one obtains

$$(*)\quad \varrho[T(z_1), T(z_2); \Delta]<A(k, \delta)\log\left[\frac{\delta\sqrt{3}}{2\varrho(z_1, z_2; D)}\right]^{-\frac{1}{4}},$$

where $A(k, \delta)$ depends only on k and δ. The metric ϱ, with respect to which the "boundary" of D is defined, is too complicated to be reproduced here, the important point being that "boundary" points correspond to prime ends. When D is the open unit disk, and $T(z)$ is a schlicht analytic function, then (*) simplifies into a result of Lavrent'ev [C.R. (Dokl.) Acad. Sci. URSS (N.S.) **4** (1936), 215–217]. *E. Reich* (Minneapolis, Minn.).

Unkelbach, Helmut. Geometrie und konforme Abbildung verallgemeinerter Kreisbogenpolygone. II. Math. Ann. **130** (1955), 327–336.

The author attempts to give a definition of angles of vertices for a generalized circular polygon introduced in part I [Math. Ann. **129** (1955), 391–414; MR **17**, 143]. It is shown that there hold certain relations between angles thus defined and coefficients of a rational function which is contained in Schwarz' differential equation satisfied by the mapping function of the half-plane onto the polygon. *Y. Komatu* (Tokyo).

Popova, N. V. A differential equation whose integrals map a half-plane onto a polygonal region. Izv. Akad. Nauk Belorussk. SSR. **1953**, 161–168. (Russian)

Conditions are studied under which the differential equation

$$\partial f(z, t)/\partial t+[\sum_{1}^{n} A_i/\{z-\lambda_i(t)\}]\partial f(z, t)/dz=0$$

is satisfied by an integral of Christoffel-Schwarz type

$$f(z, k)=c_0\int^z\prod_{\rho=1}^{n}\{z-\lambda_\rho(t)\}\prod_{k=n+1}^{m}\{z-a_k(t)\}^{\alpha_k-1}dz$$

with real $\lambda_\rho(t)$ and $a_k(t)$. These conditions form a system of differential equations for the $\lambda_\rho(t)$ and $a_k(t)$. The polygon on which the half-plane is represented remains fixed except that the cuts corresponding to $z=\lambda_\rho(t)$ increase in length with t. *A. J. Macintyre* (Aberdeen).

Cazacu, Cabiria Andreian. Le théorème d'Iversen pour des surfaces riemanniennes normalement exhaustibles. Com. Acad. R. P. Romîne **5** (1955), 1145–1150. (Romanian. Russian and French summaries)

L'auteur considère des surfaces de Riemann normalement exhaustibles comme revêtement du plan (w) par l'intermédiaire de la transformation intérieure $w=f(p)$ et démontre que cette surface possède la propriété d'Iversen si et seulement si l'ensemble des valeurs asymptotiques de f est totalement discontinu. Dans le cas simplement connexe on a une relation évidente avec les travaux d'Iverson [Thèse, Univ. de Helsingfors, 1914] et de Noshiro [Jap. J. Math. **19** (1948), no. 4, 299-327; MR **11**, 428, 872]. *L. Fourès* (Marseille).

Jurchescu, M. Au sujet des fonctions analytiques définies par des équations différentielles non algébriques. Acad. Repub. Pop. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 347–354. (Romanian. Russian and French summaries)

L'auteur démontre l'extension de la propriété d'Iversen (propriété (I) de Stoilow) aux surfaces de Riemann des fonctions analytiques $w=f(z)$ solutions d'équations différentielles $w'=\varphi(z, w)$ où φ est méromorphe en z et w. La propriété (I) a lieu relativement à la projection sur le plan (w). *L. Fourès* (Marseille).

Meier, Kurt E. Über Mengen von Randwerten meromorpher Funktionen. Comment. Math. Helv. **30** (1956), 224–233.

Let $f(z)$ be meromorphic in a region G of $\Im z>0$ whose boundary contains an interval I of the real axis. The following notations are introduced: if $0<\alpha<\pi$, $0<\beta<\pi$, $\alpha\neq\beta$, $s_x(\alpha)$ denotes the ray which emanates from the point x of the real axis and makes the angle α with the positive real axis; $w_x(\alpha, \beta)$ the angle bounded by $s_x(\alpha)$, $s_x(\beta)$; $S_x(\alpha)$ the cluster set of $f(z)$ on $s_x(\alpha)$; Γ_x the set of all values a for each of which there exist two angles α, β such that neither $S_x(\alpha)$ nor $S_x(\beta)$ contains a. The principal result asserts the existence of a set $Z\subset I$ of measure zero such that for every $x\in I-Z$ either $f(z)$ has an angular limit at x or $f(z)$ assumes every value of Γ_x infinitely often in every angle w_x. Three consequences are indicated: 1) If for a set $M\subset I$, to every $x\in M$ there exist three rays for which the corresponding cluster sets of $f(z)$ are mutually disjunct, then for almost all $x\in M$, $f(z)$ assumes every value infinitely often in every angle w_x. [By a theorem of F. Bagemihl, however, such a set M is at most enumerable, Proc. Nat. Acad. Sci. U.S.A. **41** (1955);

379–383; MR 16, 1095.] 2) If to every $x \in M$ there exist two rays $s_x(\alpha)$, $s_x(\beta)$ on which $f(z)$ is bounded, then for almost all $x \in M$, either $f(z)$ has a finite angular limit at x, or every angle w_x contains poles of $f(z)$. 3) If to every $x \in M$ there exist two rays $s_x(\alpha)$, $s_x(\beta)$ on which $f(z) \to 0$ as $z \to x$, then either $f(z) \equiv 0$ in G or for almost all $x \in M$, $f(z)$ assumes every value, with the possible exception of 0, in every angle w_x. *W. Seidel* (Notre Dame, Ind.).

Eggleston, H. G. The range set of a function meromorphic in the unit circle. Proc. London Math. Soc. (3) 5 (1955), 500–512.

Let $f(z)$ be meromorphic in $|z|<1$. If $A(n)$ denotes the annulus $1-(1/n)<|z|<1$ and $f(A(n))$ the image of this annulus under the mapping f, then the range set $R(f)$ of f is defined by the relation $R(f)=\bigcap_{n=1}^{\infty} f(A(n))$. Hence, every $R(f)$ is a G_δ set which is the intersection of a decreasing sequence of open connected sets. W. Gross [Monatsh. Math. Phys. 29 (1918), 3–47] proved, conversely, that every intersection of a decreasing sequence of open connected sets is the $R(f)$ of some function meromorphic in $|z|<1$. It was shown recently by W. Rudin [J. London Math. Soc. 30 (1955), 231–238; MR 16, 916] that there exist open sets which are not the range sets of any meromorphic function. The author gives a complete characterization of range sets of meromorphic functions in terms of the concept of the ultimate nub of a G_δ set, too complicated to be given here. This shows again the existence of open sets which are not the intersection of a decreasing sequence of open connected sets. An F_σ set K of enumerably infinite linear measure is defined such that if $R(f)$ is dense in any open set G, then $R(f)$ meets K. Here K is a fixed set for all meromorphic functions f and all open sets G. *W. Seidel* (Notre Dame, Ind.).

Oğuztöreli, M. N. Représentations intégrales de la fonction caractéristique, de la fonction de nombre et de la forme sphérique normale generalisée et extension d'un théorème de Borel. Rev. Fac. Sci. Univ. Istanbul. Sér. A. 19 (1954), 79–85. (Turkish summary)

The author gives an extension of Nevanlinna's first theorem to multiply-connected domains, exhausting by means of the level curves of the Green's function. He is evidently unaware of af Hällström's more comprehensive work on this subject [Acta Acad. Abo. 12 (1939), no. 8; MR 2, 275]. *W. K. Hayman* (Exeter).

Dalovitch, Voln. Quelques théorèmes sur les valeurs limites de la résultante de fonctions, appartenant à la classe H_δ $(0<\delta<1)$ et encore de certaines autres classes de fonctions analytiques. Bull. Soc. Math. Phys. Serbie 7 (1955), 21–38. (Serbo-Croatian. French summary)

Dalovitch, Voln. Quelques théorèmes sur l'existence des valeurs limites de la résultante de certaines classes de fonctions analytiques. C. R. Acad. Sci. Paris 242 (1956), 2087–2090.

The two papers have essentially the same content. The author tries to prove a result which is obviously false and which also occurs in his earlier note [same C.R. 241 (1955), 1441–1444; MR 14, 471]. The assertion is that $\sum_0^\infty a_\nu b_\nu z^\nu$ has radial limits at almost all points of the unit circle $|z|=1$, if $f(z)=\sum a_\nu z^\nu$ is in some class H^δ, $\delta>0$, and the real part of $g(z)=\sum b_\nu z^\nu$ is a Poisson-Stieltjes integral (counterexample: $f(z)=\sum \nu z^\nu = z(1-z)^{-2}$, $g(z)=\sum \nu^{-2} z^{2\nu}$). *A. Zygmund* (Chicago, Ill.).

Gal'perin, I. M. On a class of univalent functions. Kiev. Avtomobil.-Dorož. Inst. Trudy 2 (1955), 189–191. (Russian)

Let $f(z)=z+\cdots$ be regular and univalent in $|z|<1$ and let the image domain be convex. The author considers the class of functions of the form

$$\Phi(z)=\frac{\lambda e^{i\alpha}}{\lambda e^{i\alpha}+\mu e^{i\beta}} f(z)+\frac{\mu e^{i\beta}}{\lambda e^{i\alpha}+\mu e^{i\beta}} zf'(z)=\sum_{k=1}^{\infty} b_k z^k$$

with $\lambda, \mu>0$, and announces for such functions a number of theorems, including a representation theorem and a sharp bound on the coefficients b_k. *A. W. Goodman.*

Biernacki, M. Sur les coefficients tayloriens des fonctions univalentes. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 5–8.

Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular and univalent in $|z|<1$. The author proves that for some constant C, $|\,|a_{n+1}|-|a_n|\,| \leq C(\log n)^{3/2}$. This improves the bound $Cn^{\frac{1}{2}} \log n$ obtained earlier by Goluzin [The geometric theory of functions of a complex variable, Gostehizdat, Moscow-Leningrad, 1952; MR 15, 112]. *A. W. Goodman* (Princeton, N.J.).

Gel'fer, S. A. On the coefficient problem for p-valent functions. Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 955–958. (Russian)

Let $F_p^g(a_1, \cdots, a_m)$ denote the class of functions $f(\zeta)=\sum_{n=1}^{\infty} c_n \zeta^n$ which are regular in $|\zeta|<1$, map $|\zeta|<1$ onto a region D lying in some p-sheeted algebraic Riemann surface R of genus $\gamma \leq g$, and omit $a_1, a_2, \cdots, a_m$ in $|\zeta|<1$. Using variational formulas developed earlier by the author [same Dokl. 98 (1954), 885–888; MR 16, 459], he now proves that if $f(\zeta) \in F_p^g(a_1, \cdots, a_m)$ has maximal $|c_n|$, then $f(\zeta)$ maps $|\zeta|<1$ on the entire surface R except for a finite number of analytic arcs. *A. W. Goodman.*

Ricci, Giovanni. Sull'andamento delle funzioni maggioranti delle serie di potenze. Ann. Mat. Pura Appl. (4) 40 (1955), 285–306.

This paper is concerned primarily with relations between the two functions

$$M_2(r)=\{\sum_0^\infty |a_n|^2 r^{2n}\}^{\frac{1}{2}} \text{ and } \mathfrak{M}(r)=\sum |a_n| r^n$$

associated with the function $f(z)=\sum a_n z^n$ (which is assumed to be regular in the unit disk). First, the behavior of these functions is discussed in terms of the geometric properties of the infinite polygonal line $\mathfrak{P}(r)$ with the successive vertices $0, z_0, z_1, \cdots$, where $z_0=|a_0|$ and

$$z_n=z_{n-1}+|a_n| r^n \exp\{i(\pi/2+\arg z_{n-1})\}.$$

In this connection, the inequality $|a_k| r^k \Theta(r) \leq \mathfrak{M}(r)$ (see p. 287), where $\Theta(r)$ denotes the angle about the origin described by a point which follows the polygon $\mathfrak{P}(r)$, is incorrect, as can be seen by choosing $f(z)=1+4z$ and setting $r=1$.

Considerations regarding $\mathfrak{P}(r)$ suggest that if $\sum |a_n|=\infty$, then $\limsup_{r\to 1} \mathfrak{M}(r)/M_2(r) \geq \sqrt{2}$. The author proves this relation, and shows that it is sharp (example: $f(z)=\sum c_k z^{2^k}$, where $c_{k+1}/c_k \to \infty$).

If $\alpha(r)$ is a nonincreasing function in $0<r<1$, with $\alpha(r) \to \beta \geq 0$ as $r \to 1$, and if $M_2(r) \leq (1-r)^{-\alpha(r)}$, then

$$(1-r)^{\frac{1}{2}} \mathfrak{M}(r) \leq [1+o(1)][\alpha(r)+\tfrac{1}{2}]^{\frac{1}{2}} M_2(r),$$

$$\mathfrak{M}(r) \leq [1+o(1)] K(\beta)(1-r)^{-\alpha(r)-\frac{1}{2}},$$

where $K(\beta)=(\beta+\frac{1}{2})^{\beta+\frac{1}{2}}\beta^{-\beta}$. This includes (as the special case where $\alpha(r)$ is constant) a theorem of G. H. Hardy [Quart. J. Pure Appl. Math. **44** (1913), 147–160]. In a somewhat more complicated theorem, the assumption that $\alpha(r)$ is monotonic is abandoned. Finally, a very elaborate theorem gives estimates on $\mathfrak{M}(r)$ for the case where $\sum |a_n|^2<\infty$ and where certain other restrictions are satisfied by the Taylor coefficients a_n.

G. Piranian (Ann Arbor, Mich.).

Kahramaner, S. Sur les fonctions analytiques qui prennent la même valeur ou des valeurs données en deux points donnés (ou en n points donnés). Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 49–79. (Turkish summary)

Let $f(z)$ be regular and bounded in $|z|<1$ and satisfy $f(z_r)=\alpha$ $(r=1, \cdots, n)$. Such functions can be related in a simple way to general functions $\varphi(z)$ satisfying the hypotheses of Schwarz's lemma. This led Erim [same Rev. **12** (1947), 237–254; MR 9, 422] to obtain the exact bound for $f'(0)$. The present author discusses $f''(0)$ in special cases. He then considers functions $f(z)=\sum_0^\infty a_n z^n$ for which

$$\sum n^2(n-1)|a_n|^2<+\infty,$$

but is unable to deduce $f(z)=O(1)$ from this.

W. K. Hayman (Exeter).

Suetin, P. K. On uniqueness of series of Faber polynomials. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 423–425. (Russian)

Let K be a bounded continuum, G_∞ the complement of K, G the interior of K, and Γ the frontier of G. Let G_∞ be simply connected, and let $\{\Phi_n(z)\}$ be the Faber polynomials associated with K. For a given sequence $\{a_n\}$, the series (*) $\sum a_n\Phi_n(z)$ is formed; the question treated here is that of finding sufficient conditions in order that the series (*) be the Faber series of its sum. Two such conditions are that (*) converges uniformly on K and that Γ be a Jordan curve together with the restriction $a_n=O(\varepsilon_n/n)$, where ε_n satisfies $\sum \varepsilon_n^2/n<\infty$.

A. J. Lohwater (Ann Arbor, Mich.).

Halim, E. On the effectiveness in an open circle of simple sets of polynomials and associated sets. Proc. Math. Phys. Soc. Egypt **5** (1954), no. 2, 1–7 (1955).

The author extends the results of a preceding paper [same vol., no. 1, 31–39 (1954); MR **16**, 244] to cover effectiveness in open circles instead of closed circles. Reciprocals and products of simple sets are considered.

R. P. Boas, Jr. (Evanston, Ill.).

Hayashi, Yoshio; and Soeda, Takashi. On a geometrical viewpoint of regularity. Sci. Papers Fac. Engrg. Tokushima Univ. **3** (1951), no. 2 (new no. 4), 62–63.

Hayashi, Yoshio; and Soeda, Takashi. On a generalization of Cauchy-Riemann equations. Sci. Papers Fac. Engrg. Tokushima Univ. **3** (1951), no. 2 (new no. 4), 60–61.

Potapov, V. P. The multiplicative structure of J-contractive matrix functions. Trudy Moskov. Mat. Obšč. **4**, 125–236 (1955). (Russian)

This paper is concerned with matrix-valued functions $W(\zeta)$ of the complex variable ζ, meromorphic in $|\zeta|<1$; the values of the functions are $m\times m$ matrices with complex elements. The author's aim is to generalise to such functions the following result for ordinary complex-valued functions [see, e.g., R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1936, p. 190].

Let $w(\xi)$ be regular and bounded in $|\xi|<1$, and let $\zeta_1, \zeta_2, \cdots$ be its zeros inside the circle, repeated according to their multiplicities; then there is a real number λ and a non-decreasing real-valued function $\sigma(\theta)$, defined in $0\leq\theta\leq 2\pi$, such that, for $|\zeta|<1$,

$$w(\zeta)=e^{i\lambda}\prod_{k=1}^{\infty}\left(\frac{\zeta_k-\xi}{1-\bar{\xi}_k\zeta}\cdot\frac{|\xi_k|}{\xi_k}\right)\exp\left\{\int_0^{2\pi}\frac{\xi+e^{i\theta}}{\xi-e^{i\theta}}d\sigma(\theta)\right\}.$$

The function $w(\xi)$ is thus decomposed into a Blaschke product over its zeros inside the circle and an exponential integral that takes account of its behaviour near the circumference. More generally, if $w(\xi)$ is meromorphic and can be expressed as the quotient of two bounded functions, the formula will contain two Blaschke products, one for the zeros and one for the poles, and $\sigma(\theta)$ will be a function of bounded variation on $(0, 2\pi)$.

To describe the author's main theorem, we require some definitions. Let

$$J=\begin{bmatrix} I_p & O \\ O & -I_q \end{bmatrix},$$

where $p+q=m$; the matrix J is kept fixed throughout. The matrix W is said to be J-contractive if $WJW^*\leq J$, i.e., $J-WJW^*$ is a non-negative definite Hermitian matrix, and U is said to be J-unitary if $UJU^*=J$. Suppose now that $W(\xi)$ is meromorphic in $|\xi|<1$, and J-contractive for each ζ for which it is defined, and that $\det W(\xi)$ does not vanish identically. Then the main theorem states that, for $|\xi|<1$,

$$W(\xi)=B_\infty(\xi)B_0(\xi)\hat{\int}\exp\left\{\frac{\zeta+e^{i\vartheta(t)}}{\zeta-e^{i\vartheta(t)}}\mathrm{d}E(t)\right\},$$

where $B_\infty(\xi)$ and $B_0(\xi)$ are generalised Blaschke products, and the integral is a multiplicative Stieltjes integral analogous to the multiplicative integrals of Volterra and Schlesinger [see, e.g., V. Volterra and B. Hostinský, Opérations infinitésimales linéaires, Gauthier-Villars, Paris, 1938, chap. VI]. The product $B_\infty(\xi)$ is of the form

$$\prod_{k=1}^{\infty}U_k\begin{bmatrix} I & O \\ O & \dfrac{1-\bar{\mu}_k\xi}{\mu_k-\xi}\dfrac{\mu_k}{|\mu_k|}I_{q_k} \end{bmatrix}U_k^{-1},$$

where μ_k runs over the poles of $W(\xi)$ in $|\xi|<1$, $q_k\leq q$ for all k, and each matrix U_k is J-unitary; similarly,

$$B_0(\xi)=\prod_{k=1}^{\infty}V_k\begin{bmatrix} \dfrac{\lambda_k-\xi}{1-\bar{\lambda}_k\xi}\dfrac{\lambda}{|\lambda_k|}I_{p_k} & O \\ O & I \end{bmatrix}V_k^{-1},$$

where λ_k runs over the zeros of $\det[B_\infty^{-1}(\xi)W(\xi)]$ in $|\xi|<1$, $p_k<p$ for all k, and each V_k is J-unitary. The multiplicative integral is defined as a limit of products of the form

$$\prod_{r=1}^{n}\exp\left\{\frac{\zeta+e^{i\vartheta(t_r)}}{\zeta-e^{i\vartheta(t_r)}}[E(t_r)-E(t_{r-1})]\right\},$$

where $\vartheta(t)$ is a non-decreasing real function of t, $E(t)J$ is a non-decreasing Hermitian matrix-function, and $\operatorname{tr}[E(t)J]=t$.

These results have been an essential tool in the work of M. S. Livšic on the spectral theory of non-self-adjoint operators in Hilbert space [Mat. Sb. N.S. **34(76)** (1954), 145–199; MR **16**, 48]. The author also indicates appli-

cations to an inverse problem for systems of linear ordinary differential equations of the form

$$\frac{dy_j}{dt}=i\lambda\sum_{k=1}^{m}y_k h_{kj}(t)\quad(1\leq j\leq m,\ 0\leq t\leq l),$$

where the matrix $H(t)J$ is Hermitian and non-negative definite. Writing the system in matrix form as

$$\frac{dy}{dt}=i\lambda YH(t),$$

the author denotes by $Y(t,\lambda)$ the matrix solution with the initial condition $Y(0,\lambda)=I$, where I is the identity matrix. He shows that if $W(\lambda)$ is an integral matrix-function of λ, J-contractive in the upper half-plane and J-unitary on the real axis, then there is a system of the above form such that $W(\lambda)=Y(l,\lambda)$. These results can be used for finding a system having a given spectrum under given boundary conditions.

The paper contains many detailed results concerning J-contractive matrices; for instance, the most general analytic (1, 1) mapping of the set of I-contractive matrices onto the set of J-contractive matrices is determined, and expressed in terms of fractional linear mappings of the form

$$Y=(AX+B)(BX+A)^{-1},$$

and generalisations are found for the lemmas of Schwarz and Julia.

The proof of the main theorem is long and complicated, the chief difficulties being concerned with the process of detaching one by one the elementary factors in the Blaschke products, and in proving that the products are convergent.

An appendix is devoted to the definition and elementary properties of the multiplicative Stieltjes integral, including a generalisation of Helly's theorem.

F. Smithies (Cambridge, England).

See also: Šidlovskiĭ, p. 947; Lehto, p. 960; Gaier, p. 960; Cowling, p. 961; Geronimus, p. 962; Haplanov, p. 972.

Harmonic Functions, Potential Theory

★**de Rham, G. Sur certaines équations de la théorie des formes différentielles harmoniques.** Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954, pp. 67–82. Georges Thone, Liège; Masson & Cie, Paris, 1955.

The author considers the Dirichlet and Neumann problems for elliptic equations given by $(\delta d+c_1,\ d\delta+c_2)\mu=\beta$, where the exterior differential form β, and constants $c_1\geq 0$, $c_2>0$ are given, and d and δ are the exterior differential and codifferential operators. It is assumed that the equation is defined in a Riemannian manifold V of class C^∞. If V were a subdomain of a manifold W, and the boundary B of V were smooth in W, then the currents $F\alpha$ and $F'\alpha$ given by $\int_B \alpha\wedge *\beta$ and $\int_B \beta\wedge *\alpha$ completely determine the coefficients of α at all points of B. Since

$$F\alpha(\beta)=\int_V S_V d\alpha\wedge *\beta-\int_V \alpha\wedge *\delta\beta=(d\alpha,\beta)-(\alpha,\delta\beta)$$

by Stokes' theorem, and analogously for $F'\alpha$, the Dirichlet and Neumann problems can be intrinsically formulated in V, making regularity assumptions on the boundary of V unnecessary. The results obtained extend those of G. F. D. Duff and D. C. Spencer [Ann. of Math. (2) **56** (1952), 128–156; MR **13**, 987]. $E(\alpha,\beta)=(d\alpha,d\beta)+c_1(\delta\alpha,\delta\beta)+c_2(\alpha,\beta)$ serves to define an inner product in a pre-Hilbert space consisting of forms of class C^∞ for which $E(\alpha,\alpha)<\infty$. If $c_1>0$, $(\beta,\beta)<\infty$ and α belongs to the completed Hilbert space H, then there exists μ such that $E(\mu,\varphi)=(\beta,\varphi)$ for all $\varphi\in H$, and $F\mu=F\alpha$, $F'\mu=F'\alpha$, or "$\mu=\alpha$" on the "boundary" of V. μ is a regular solution of (1) if both α and β are sufficiently regular. If $c_1=0$, then only $F\mu=F\alpha$ can be prescribed for Dirichlet's problem or $F'd\mu=F'd\alpha$ for the Neumann problem. The proofs combine the method of orthogonal projection with the theory of currents.

A. N. Milgram.

Morrey, Charles B., Jr.; and Eells, James, Jr. A variational method in the theory of harmonic integrals. Proc. Nat. Acad. Sci. U.S.A. **41**, 391–395 (1955).

The authors investigate the theory of harmonic integrals (on compact Riemannian manifolds) with a variational method which is applicable to a much lower differentiability class than has previously been considered. It is necessary only that the functions which give the coordinate transformations of the manifold have first derivatives which satisfy uniform Lipschitz conditions.

A family of topologically equivalent inner products of r-forms is introduced; the inner products depend (among other things) on the derivatives of the components of the forms. $\mathfrak{B}_2{}^r$ denotes the class obtained when the Lipschitz r-forms are completed with respect to (any) one of these inner products.

The differential operators d and δ are bounded operators from $\mathfrak{B}_2$ to $\mathfrak{L}_2$. Weak convergence in $\mathfrak{B}_2{}^r$ implies strong convergence in $\mathfrak{L}_2{}^r$. The Dirichlet integral $(d\omega,d\omega)+(\delta\omega,\delta\omega)$ is lower semicontinuous with respect to weak convergence in $\mathfrak{B}_2{}^r$.

From the variational method is obtained the finite dimensionality of the harmonic fields, the existence of a form called the "potential" ($\Delta^{-1}\omega_0$ if there is sufficient differentiability) of any form ω_0 orthogonal to the harmonic fields, and a decomposition theorem in which the exact nature and differentiability of the projections are closely examined. Differentiability properties of potentials and harmonic fields are studied.

M. P. Gaffney.

Oleĭnik, O. A. On the stability of the Neumann problem. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 223–225. (Russian)

Let D be a plane region bounded by a smooth curve Γ, and suppose that there is a constant $\delta>0$ such that to each point P of Γ there corresponds a closed disc containing P and lying in $D\cup\Gamma$. The class of all functions u continuous on $D\cup\Gamma$, harmonic on D, and admitting normal derivatives on Γ is denoted by T. A concise proof is given for the following theorem: there exists a constant K (determined by D) such that to each $\varepsilon>0$ and each pair u_1, u_2 of functions in T satisfying

$$\left|\frac{\partial u_1}{\partial n}-\frac{\partial u_2}{\partial n}\right|\leq\varepsilon \text{ on } \Gamma$$

there corresponds a constant C yielding

$$|u_1-u_2-C|\leq K\varepsilon \text{ on } D.$$

M. G. Arsove (Seattle, Wash.).

Slobodeckiĭ, L. N.; and Babič, V. M. On boundedness of the Dirichlet integrals. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 604–606. (Russian)

Let f be a measurable function on $(-\infty,\infty)$ with period

2π, and let

$$\omega(h, f)=\Big[\int_0^{2\pi}|f(\theta+h)-f(\theta)|^2d\theta\Big]^{\frac{1}{2}}.$$

Then the condition

$$\int_0^{2\pi}|f(\theta)|^2d\theta+\int_0^1\Big[\frac{\omega(h, f)}{h}\Big]^2dh<+\infty$$

is necessary and sufficient for the existence of a function u on the unit circle having finite Dirichlet integral and satisfying $\lim_{r\to 1}u(re^{i\theta})=f(\theta)$ [mean square]. The author proves this result, along with certain generalizations on the domain of u and on the Dirichlet integral.

M. G. Arsove (Seattle, Wash.).

Ėĭdus, D. M. Estimates on the derivatives of Green's function. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 207–209. (Russian)

If G is the Green's function for a region Ω, then there exist constants c_k (determined by Ω) such that

$$\left|\frac{\partial G(x, y)}{\partial x_k}\right|\leqq\frac{c_k}{r_{xy}^{n-1}},$$

where x_k is the kth coordinate ($k=1, 2, \cdots, n$) of x and r_{xy} is the distance between x and y. Stating that the proof for $n=3$ given by P. Lévy [Acta Math. **42** (1919), 207–267, p. 259] seems in error, the author presents a proof based on the methods of integral equations. The boundary of Ω is assumed to satisfy the condition of Liapounoff.

M. G. Arsove (Seattle, Wash.).

Lehto, Olli. On the first boundary value problem for functions harmonic in the unit circle. Ann. Acad. Sci. Fenn. Ser. A. I. no. **210** (1955), 26 pp.

Let $f(z)$ denote any single-valued function of the complex variable z in $|z|<1$. If $\lim_{r\to 1}f(re^{i\theta})$ exists for almost all values of θ in $0\leqq\theta<2\pi$, this limit is denoted by $f(e^{i\theta})$. Making use of a recent result of Bagemihl and Seidel [Math. Z. **61** (1954), 186–189; MR **16**, 460], it is shown that if $\psi_1(\theta)$, $\psi_2(\theta)$ are arbitrary measurable functions, real-valued in the extended sense, in $(0, 2\pi)$, then there exists a function $f(z)$, holomorphic in $|z|<1$, such that $f(e^{i\theta})=\psi_1(\theta)+i\psi_2(\theta)$ almost everywhere in $(0, 2\pi)$. It should be noted that this result is a special case of a theorem of Bagemihl and Seidel [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 740–743; MR **17**, 249]. In particular, given any measurable $\psi(\theta)$, there exists a harmonic function $u(z)$ in $|z|<1$ such that $u(e^{i\theta})=\psi(\theta)$ almost everywhere [cf. W. Kaplan, Michigan Math. J. **3** (1955), 43–52; MR **17**, 31]. The author next studies the explicit construction of harmonic functions $u(z)$ with prescribed boundary values $\psi(\theta)$ which are finite almost everywhere. In this case it is always possible to find a holomorphic $F(z)$ for which $F(e^{i\theta})$ is real and finite almost everywhere and $\psi(\theta)/F(e^{i\theta})$ belongs to L_1. This part is related to an earlier investigation of O. Lokki [Ann. Acad. Sci. Fenn. Ser. A. I. no. **202** (1955); MR **17**, 725]. An explicit expression for a solution $u(z)$ of the problem is then

$$u(z)=\frac{1}{2\pi}\operatorname{Re}\Big\{F(z)\int_0^{2\pi}\frac{\psi(\theta)}{F(e^{i\theta})}\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\Big\}.$$

The method is illustrated by constructing solutions $u(z)$ for certain special $\psi(\theta)$ such as $1/\theta$, $[\sin(1/\theta)]^{-2}$. It is shown how to construct for any given $\psi(\theta)$ a solution $u(z)$ whose modulus on a prescribed radius is either of arbitrarily rapid growth or of a given finite order. The case that $\psi(\theta)$ is infinite on a set of positive measure is considered next. Two special cases are treated. A harmonic function $u(z)$ is constructed such that on a given arc A on $|z|=1$, $u(e^{i\theta})=\infty$ almost everywhere on A and $u(e^{i\theta})$ is finite almost everywhere on the complementary arc. A similar result is obtained when the arc is replaced by an arbitrary non-void open set on $|z|=1$.

W. Seidel.

Suharevskiĭ, I. V. On some problems of the theory of the logarithmic potential. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 426–429. (Russian)

Announcement, without proofs, of results on certain boundary-value problems of potential theory, related to the modified Dirichlet problem studied by N. I. Mushelišvili [Singular integral equations..., OGIZ, Moscow-Leningrad, 1946; MR **8**, 586; **11**, 523]. The principal result is applied to the solution of certain problems which generalize the classical Dirichlet and Neumann problems.

W. Seidel (Notre Dame, Ind.).

See also: Cooke, p. 977.

Series, Summability

Kolberg, O. Über ein Problem von Viggo Brun. Math. Scand. **3** (1955), 221–223 (1956).

The problem is to sum the series $\sum_0^\infty n^n(n+2)^{-n-2}$, where $0^0=1$. The author corrects the answer $\frac{1}{2}$ given by Halvorsen [Norsk Mat. Tidsskr. **34** (1952), 95–96] to $\frac{1}{2}-\int_0^\infty x^x(x+2)^{-x-2}dx$. The modified answer showed up an error in an inversion formula of V. Brun [see the following review].

J. Korevaar (Madison, Wis.).

Brun, Viggo. Une formule d'inversion corrigée. Math. Scand. **3** (1955), 224–228 (1956).

Let $a(x)=x+\sum_2^\infty a_nx^n$, where the power series has positive radius of convergence, and let $a_r(x)=a(a_{r-1}(x))$, $a_0(x)=x$. Tambs Lyche expressed the coefficients A_n of the inverse function $A(x)=x+\sum_2^\infty A_nx^n$ of $a(x)$ in terms of the coefficients of the functions $a_r(x)$ [Bull. Soc. Math. France **55** (1927), 102–113]. The author uses this result to show that

$$A(x)=S(x)+R(x),$$

$$S(x)=x-x^2a_1''(x)/2!+x^3a_2'''(x)/3!-\cdots,$$

$$R(x)=\lim_{m\to\infty}\frac{1}{m!}\int_0^x u^m\sum_2^m\binom{m}{k}\Big(\frac{-x}{u}\Big)^k a_{k-1}^{(m+1)}(x-u)du,$$

whenever the series for $S(x)$ converges. In a previous paper [C. R. Acad. Sci. Paris **194** (1932) 2276–2277] the author has assumed that $R(x)$ is always zero.

J. Korevaar (Madison, Wis.).

Dux, Erich. Ein kurzer Beweis der Divergenz der unendlichen Reihe $\sum_{r=1}^\infty\frac{1}{p_r}$. Elem. Math. **11** (1956), 50–51.

Gaier, Dieter. Über die Äquivalenz der $|B_k|$-Verfahren. Math. Z. **64** (1956), 183–191.

The series (1) $\sum a_n$ with partial sums s_n is B-summable to s provided its transform $f(x)=e^{-x}\sum s_nx^n/n!$ exists for $x>0$ and tends to s as $x\to\infty$; it is $|B|$-summable if in addition the total variation of $f(x)$ on $0\leqq x<\infty$ is finite. The series (1) is B_k-summable provided it can be made B-

summable by adjoining k zeros, at the beginning; the definition of $|B_k|$-summability is similar.

All the methods B_k are equivalent in the space of series (1) for which there exists a constant K such that $a_n=O(K^n)$; and they are not equivalent in the space of series for which $a_n=O(K^n n^{\varepsilon n})$ ($\varepsilon>0$). The author now extends this theorem to the methods $|B_k|$. In order to establish his result, he proves extensions of theorems on functions regular in the half-plane. Example: If $f(z)$ is regular and of exponential type, in $\Re z\geqq 0$, and if $\int_0^\infty |f(x)|^p dx<\infty$, for some $p>0$, then $\int_0^\infty |f'(x)|^p dx<\infty$. Moreover, the proposition is not true for entire functions of order $1+\varepsilon$ ($\varepsilon>0$).

G. Piranian (Ann Arbor, Mich.).

Petersen, G. M. Sequences of 0's and 1's and Toeplitz methods of summability. Amer. Math. Monthly **63** (1956), 174–175.

The author shows that for each regular Toeplitz matrix A there exists a sequence of 0's and 1's, containing no triplet of consecutive 0's or 1's, which is not summable by A. Proof: Otherwise A would sum all sequences of this type to the same value; this leads to a contradiction with the properties "linear" and "regular" of A.

K. Zeller (Tübingen).

Šteglov, M. P. On a generalization of Vijayaraghavan's Tauberian theorems. Ukrain. Mat. Ž. **7** (1955), 333–338. (Russian)

Let

$$s_n=\sum_{\nu=0}^{n} a_\nu,\quad \sigma_n=(1/n)\sum_{\nu=0}^{n} s_\nu,\quad \varphi(u)=\sum_{n=0}^{\infty} a_n e^{-nu}$$

and $d_s=\liminf s_n$, $d_\sigma=\liminf \sigma_n$, $d_\varphi=\liminf \varphi(u)$ for $n\to\infty$ or $u\to\infty$; let further D_s, D_σ, D_φ denote the corresponding upper limits. The problem is to establish relations between the d's and the D's of different kind, under the assumption that the series satisfies certain one-sided Tauberian conditions. The main results are the following (1) $\sup (d_\sigma-d_s)=\sup (d_\varphi-d_s)=C$, where the supremum is taken for all series with $a_n<C/n\log\log n$; (2) $d_\sigma=d_\varphi$ if $a_n<o(1/n)$; (3) $\sup (D_s-D_\sigma)=C$ for all series with $a_n<C/n$; (4) if $a_n<C/n$, then d_σ and d_φ are either both finite or both infinite of the same sign. Among special cases are some former results of the author [Dokl. Akad. Nauk SSSR (N.S.) **87** (1952), 697–700; Ukrain. Mat. Ž. **5** (1953), 299–303; MR **14**, 1078; **15**, 306] and of Vijayaraghavan [J. London Math. Soc. **2** (1927), 215–222]. The proofs are by a direct construction of extremal series for which the supremum in question is attained.

G. G. Lorentz.

Vučković, Vladeta. Deux théorèmes de type mercerien. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 53–58.

Let $\sigma_n=s_{n-1}+c(s_n-s_{n-1})$, $\Re c>\frac{1}{2}$. The author proves that if σ_n is Abel or Nörlund summable to σ, then so is s_n. This is one of the many generalizations of a remark by Pólya [Arch. Math. Phys. (3) **24** (1915), 282] that under the same assumptions, convergence of σ_n implies that of s_n.

G. G. Lorentz (Detroit, Mich.).

Šteglov, M. P. Solution of certain extremal problems in the theory of divergent series. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 703–704. (Russian).

Let $W=W(C,\omega)$ denote the class of real series $\sum_{n=0}^{\infty} a_n$ with bounded partial sums s_n for which $a_n<C/n$ and $\omega=\limsup s_n-\liminf s_n$, for fixed positive parameters C and ω. Various extreme values of the differences of the upper (and lower) limits for the partial sums and those of the $(C,1)$ means σ_n and the Poisson transforms $\phi(u)=\sum a_n e^{-nu}$ are determined explicitly in terms of C and ω, and their asymptotic properties listed. We quote only the following:

$$\sup_W \{\liminf \sigma_n-\liminf s_n\}=C\log\frac{C+\omega}{C},$$

$$\sup_W \{\liminf \phi(u)-\liminf s_n\}=(1-e^{-\lambda})+C\int_\lambda^{\lambda e^{\omega/C}} (xe^x)^{-1}dx,$$

λ being the positive root of $\exp x(1-e^{-\omega/C})=1-\omega x/C$. Corresponding results for the class W_+ (where the condition $a_n<C/n$ is replaced by $|a_n|<C/n$) are also given, and several open questions are cited.

G. Klein.

Cowling, V. F. On the Euler summability of a class of Dirichlet series. Tôhoku Math. J. (2) **7** (1955), 240–242.

Die Arbeit beschäftigt sich mit der E_r-Summierung ($0<r<1$) von Dirichletschen Reihen (1) $\sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$ mit $\lambda_n=(n+1)^\alpha$ ($0<\alpha<1$). Es bezeichne $\sum_{n=0}^{\infty} A_n$ die E_r-Transformierte von (1). Satz: Konvergiert $\sum_{n=0}^{\infty} A_n r^{-n}$ für ein r_0 in $0<r<1$ und ein z_0, so ist die Reihe (1) $|E_r|$-summierbar für $r>r_0$ und alle z mit $|\arg(z-z_0)|<\pi(1-\alpha)/2$.

D. Gaier (Stuttgart).

Karadžić, Lazar. Quelques théorèmes sur les séries de fonctions. Bull. Soc. Math. Phys. Serbie **7** (1955), 165–170 (1956). (Serbo-Croatian summary)

In the present paper the author continues his work on an application of the geometry of Lobatschewsky to the convergence of infinite series. The following theorem is typical of those proven. If

$$\limsup_n (\exp\{\sum_1^n |\alpha_k(x)|\}|\beta_n|)\leqq M,$$

where M is independent of $x\in[a,b]$, then $\sum \alpha_n(x)\beta_n(x)$ converges uniformly for $x\in[a,b]$.

V. F. Cowling.

Copping, J. Transformations of multiple sequences. Proc. London Math. Soc. (3) **6** (1956), 224–250.

Denote by $x=\{x_i\}$ a sequence of complex numbers of multiplicity γ (γ a positive integer) where i stands for a set of γ suffixes. Let g_v (for $v=1,2,\cdots,2^\gamma-1$) denote one of the different subsets of suffixes selected from i (excluding the null set) and denote by h_v the complement of g_v with respect to i so that for any v ($\leqq 2^\gamma-1$) x_i may be written x_{g_v,h_v}. If $\lim_{g_v\to\infty} x_{g_v,h_v}=y_{h_v}$ exists (as all suffixes in g_v tend to infinity independently) and is finite for each fixed set of values of h_v, then x is called partially convergent with respect to y_v and one writes $x\in c(g_v)$. If the limits are zero, one writes $x\in C_0(g_v)$. In the present paper the author finds (among other things) necessary and sufficient conditions in order that a given multiply infinite matrix $(a_{n,i})$ transform every sequence $x\in C_0(g_v)$ into a sequence $z=\{z_n\}\in C_0(l)$, where n denotes a set of suffixes and l any given subset of n. The results obtained extend some related results of Hamilton [Duke Math. J. **2** (1936), 29–60].

V. F. Cowling (Lexington, Ky.).

Rajagopal, C. T. Additional note on some Tauberian theorems of O. Szász. Pacific J. Math. **5** (1955), 971–975.

Contrairement au procédé appliqué dans la démonstration du théorème principal, dans sa note antérieure [même J. **2** (1952), 377–384; MR **14**, 265] l'auteur utilise ici une voie inverse. Il montre que, du fait que la série

$\sum_1^\infty a_n$ est (ϕ, λ)-sommable vers s avec $\lambda_n \uparrow \infty$ et $\lambda_{n+1}/\lambda_n \to 1$, sous la condition supplémentaire du type tauberien

$$\liminf (1/\lambda_n) \sum_{n+1}^{\infty} \lambda_\nu a_\nu \geqq 0 \quad (n \to \infty),$$

résulte que $\sum_1^\infty a_k$ converge vers s. Ce résultat est une simple conséquence du corollaire III′ de la note citée, et qui se trouve dans cette même note, à la fin de la déduction. Ce corollaire sert ici comme un lemme préliminaire. Un résultat semblable est donné pour les intégrales.

M. Tomić (Beograd).

See also: van der Corput and Hefflinger, p. 949; Haplanov, p. 972.

Interpolation, Approximation, Orthogonal Functions

Džvaršeišvili, A. G. An inequality of S. N. Bernšteĭn in the space of (D^*) integrable functions and its applications. Soobšč. Akad. Nauk Gruzin. SSR **16** (1955), 257–262. (Russian)

Let $D=\{f(x)\}$ be the space of functions integrable in the sense of Denjoy-Perron in $(0, 2\pi)$ and with the norm $\|f\|=\sup_{g\in V}\{|\int_0^{2\pi} f(x)g(x)dx|\}$, where V is the set of all functions $g(x)$ defined in $(0, 2\pi)$ such that $|g(x)|\leqq 1$, $V_0^{2\pi}[g]\leqq 1$, and let $\omega(f, \delta)=\sup_{0\leq h\leq\delta}\|f(x+h)-f(x)\|$. If $H_n=\{T(x)\}$ are the trigonometrical polynomials of degree $\leqq n$, and $E_n(f)=\inf_{T\in H_n}\{\|f(x)-T(x)\|\}$, then: 1) $\|T'(x)\|\leqq 2n\pi^{-1}\|T(x)\|$, for any $T\in H_n$. 2) $E_n(f)\leqq 12\omega(f, n^{-1})$. 3) If $E_n(f)\leqq An^{-\alpha}$ $(0<\alpha<1)$, then $\|f(x+h)-f(x)\|\leqq Ch^\alpha$ $(h>0)$.

M. Cotlar (Mendoza).

Riney, T. D. On the coefficients in asymptotic factorial expansions. Proc. Amer. Math. Soc. **7** (1956), 245–249.

The author obtains an explicit recursion formula for the coefficients in the asymptotic factorial expansion (in a right half-plane) of the function

$$\prod_{i=1}^{p} \Gamma(\omega+\sigma_i) / \prod_{j=0}^{q} \Gamma(\omega+\varrho_j),$$

where p and q are non-negative integers with $p\leqq q$, σ_i and ϱ_j are arbitrary complex parameters, and ω is a complex variable.

T. E. Hull (Vancouver, B.C.).

Geronimus, Ya. L. On asymptotic properties of orthogonal polynomials. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 175–178. (Russian)

Désignons par $\{P_n(z)\}_0^\infty$, $P_n^*(z)=z^n\overline{P}_n(1/z)$ les polynômes orthonormaux sur le cercle $z=e^{i\theta}$ $(0\leqq\theta\leqq 2\pi)$, par rapport à une fonction de poids $p(\theta)$ bornée presque partout dans $[0, 2\pi]$ [voir G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939, p. 280; MR 1, 14], par $\omega_2(\delta, p)$ le module de continuité intégral de $p(\theta)$ dans $L_2(0, 2\pi)$ et par $\pi(z)$ l'expression

$$\pi(z)=\exp\left\{-\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log p(\theta)d\theta\right\},\ |z|<1,$$

qui est une fonction holomorphe et $\neq 0$ dans $|z|<1$. L'auteur démontre que, de $\omega_2(\delta, 0)=o(\sqrt{\delta})$, l'existence d'une des limites $\lim P_n^*(e^{i\theta})$, $n\to\infty$, $\lim \pi(re^{i\theta})$, $r\to 1-0$, entraîne l'autre. Si $p(\theta)$ est une fonction à variation bornée dans $[0, 2\pi]$, continue dans $[\alpha, \beta]\subset[0, 2\pi]$ avec le module de continuité $\omega(x, p)$ satisfaisant à la condition (*) $\int_0^\delta \omega(x, p)x^{-1}dx<\infty$, alors, dans $\alpha+\varepsilon\leqq\theta\leqq\beta-\varepsilon$, $\varepsilon>0$, on a uniformément $\lim P_n^*(e^{i\theta})=\pi(e^{i\theta})$, $n\to\infty$. Si $[\alpha, \beta]=[0, 2\pi]$ et si (*) se réduit à la condition de Dini-Lipschitz, le dernier résultat se ramène à un théorème de S. Bernstein et G. Szegö [voir le livre de G. Szegö, cité plus haut, § 12.4].

M. Tomić (Beograd).

Dieulefait, Carlos E. Sulla legge di distribuzione degli zeri dei polinomi ortogonali classici di grado n, considerata al divergere di n. Giorn. Ist. Ital. Attuari **17** (1954), 36–46.

Arguing from a vague analogy between electrostatic equilibrium and stochastic equilibrium the author gives formal calculations leading to the asymptotic distribution of the zeros of, for example, Legendre's polynomials. For these (and, more generally, for the Jacobi polynomials) his results happen to be correct, but are known [cf. G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939, Theorem 8.9.1, p. 232; MR 1, 14]. When applied to a similar problem that has been investigated by C. A. Rogers [Acta Math. **82** (1950), 185–208, Lemmas 1–6; MR **11**, 501] the author's method gives an incorrect result.

H. P. Mulholland (Birmingham).

Shapiro, Victor L. The symmetric derivative on the ($k-1$)-dimensional hypersphere. Trans. Amer. Math. Soc. **81** (1956), 514–524.

The Gegenbauer polynomials P_n^α, $\alpha>-1$, are defined by $(1-2rt+r^2)^{-\alpha}=\sum_0^\infty r^n P_n^\alpha(t)$. If μ is a totally-finite signed Borel measure on the unit $(k-1)$-dimensional hypersphere Ω of real Euclidean k-space, set $\lambda=(k-2)/2$ and define

$$Y_n(x; d\mu)=\frac{(n+\lambda)\Gamma(\lambda)}{2\pi^{\lambda+1}}\int_\Omega P_n^\lambda[(x, y)]d\mu(y),$$

(x, y) denoting the ordinary inner product in Euclidean k-space, and $S(x; d\mu)=\sum_0^\infty Y_n(x; d\mu)$. If $D(x_0, h)=\{x\in\Omega; (x, x_0)\geqq\cos h\}$, let $|D(x_0, h)|$ denote the $(k-1)$-dimensional volume of $D(x_0, h)$ and set $\mu_s(x_0)=\lim_{h\to 0}\mu[D(x_0, h)]/|D(x_0, h)|$ whenever this limit exists. The author proves: (i) If $\mu_s(x_0)$ exists as a real number, then $S(x_0; d\mu)$ is (C, α)-summable to $\mu_s(x_0)$ for $\alpha>\max[3/2, k-2]$. (i′) The conclusion of (i) is false if $\alpha=k-2$. (ii) If $\mu_s(x_0)$ exists (real) and $|\mu|[D(x_0', \varepsilon)]=0$ for some $\varepsilon>0$, $|\mu|$ being the total variation of μ and x_0' being the point on Ω diametrically opposite x_0, then $S(x_0; d\mu)$ is (C, δ)-summable to $\mu_s(x_0)$ for $\delta>k/2$. (ii′) The condition $|\mu|[D(x_0', \varepsilon)]=0$ in (ii) can be replaced by the condition that $\mu(E)=\int_E f(y)d\Omega(y)$ for some Ω-integrable f and

$$\int_{(D(x_0', \varepsilon))}\{|f(y)|[1-(x_0', y)^2]^{-\frac{1}{2}}\}d\Omega(y)<\infty.$$

(iii) If $\mu(E)=\int_E f(y)d\Omega(y)$ for some Ω-integrable f and $Y_n(x_0; d\mu)=O(1/n)$, then $S(x_0; d\mu)=\beta$ (real) if and only if $\mu_s(x_0)=\beta$.

(i) is a generalization of a well-known result for Fourier-Stieltjes series [A. Zygmund, Trigonometrical series, Warsaw-Lvov, 1935, p. 55] and has been proved for μ absolutely continuous with respect to Ω at the points of continuity of $d\mu/d\Omega$ by L. Koschmeider [Math. Ann. **104** (1931), 387–402]. (ii′) for f continuous at x_0 is due to K. K. Chen [Sci. Rep. Tôhoku Imp. Univ. **17** (1928), 1073–1089] and has been proved by Koschmeider [reference above] under the hypothesis

$$\lim_{h\to 0}\int_{D(x_0, h)}\frac{|f(y)-\mu_s(x_0)|}{|D(x_0, h)|}d\Omega(y)=0,$$

a result stronger than Chen's but weaker than (ii′).

A. E. Livingston (Seattle, Wash.).

Grebenyuk, D. G. Construction of polynomials of several variables deviating least from a given function. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **15** (1955), 111–121. (Russian)

The author considers methods for constructing polynomials in several variables deviating least from a given function. He limits his discussion to the case of two variables. The treatment is based on the author's earlier paper [same Trudy **10** (1953), 105–128], and the notations used are with reference to that paper. *S. Kulik.*

Freid [Freud], Geza. One-sided L_1-approximations and their application to theorems of Tauberian type. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 689–691. (Russian)

This is a resumé of Acta Sci. Math. Szeged **16** (1955), 12–28 [MR **17**, 30]. *G. Klein* (Cambridge, Mass.).

See also: Mel'nikov, p. 966; de Castro, p. 967; Chatterjee, p. 967; De Claris, p. 1030.

Trigonometric Series and Integrals

Izumi, Shin-ichi. Some trigonometrical series. XIV. Proc. Japan Acad. **31** (1955), 324–326.

Let $f(t)$ belong to $L_p(0, 2\pi)$ $(p \geqq 1)$ and let

$$m_p(h) = \max_{0 \leq x \leq 2\pi} \int_0^h |f(x+t)|^p dt$$

and

$$\omega_p(h) = \max_{|t| \leq h} \int_0^{2\pi} |f(x+t)-f(x)|^p dx.$$

The author proves using the Poisson integrals of $f(t)$ that if $f(t)$ is not constant then $m_p(h) \leq A\omega_p(h)/h^{p-1}$. For $p=1$, this is contained in a paper of E. Hille and the reviewer [Duke Math. J. **21** (1954), 587–591; MR **16**, 241], where it is established less directly using interpolation theory. *G. Klein* (Cambridge, Mass.).

Biernacki, M. Sur les zéros des polynômes trigonométriques dont la suite des coefficients est monotone. Ann. Polon. Math. **1** (1955), 380–387.

The author studies the zeros of the trigonometric polynomials $p_k(\theta) = a_k \sin k\theta + \cdots + a_n \sin n\theta$, $q_k(\theta) = a_k \cos k\theta + \cdots + a_n \cos n\theta$ $(k<n)$ in which all the a_j are positive. He shows that if

$$a_{j+1}/a_j \leq \{k/(k+1)\}^k \{(j+k-1)/(j+k)\} \quad \text{for } j = k, k+1, \cdots, n-1,$$

then p_k and q_k have each exactly $2k$ zeros on the interval $0 \leq \theta < 2\pi$, each zero being simple. In the case that all the zeros of the polynomial $a_k + a_{k+1}z + \cdots + a_n z^{n-k}$ are real, he shows that, if

$$ka_k \geqq (k+1)a_{k+1} \geqq \cdots \geqq na_n > 0$$

with at least one equal sign not holding, then p_k and q_k have each exactly $2k$ zeros on the interval $0 \leqq \theta < 2\pi$ and, in particular, $p_k(\theta) > 0$ for $0 < \theta \leqq \pi/2k$ and $p(\pi/k) < 0$. These results are intended to be analogous to those due to Pólya and Szegö [G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., v. 23, New York, 1939, pp. 130–131; MR **1**, 14] that $p_1(\theta)$ and $q_0(\theta)$ have each exactly $2n$ real zeros on $0 \leqq \theta < 2\pi$ and that $q_0(\theta)$ has a single simple zero in each interval

$$\{(2s-1)/(2n+1)\}\pi < \theta < \{(2s+1)/(2n+1)\}\pi, \quad s+1, 2, \cdots, 2n.$$

M. Marden (Milwaukee, Wis.).

Aljančić, S.; Bojanić, R.; et Tomić, M. Sur l'intégrabilité de certaines séries trigonométriques. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 67–84.

Let $f(x)$ be a sine or cosine series, with coefficients λ_n ultimately decreasing to zero. The authors generalize known results on the integrability of $x^{-\gamma}f(x)$ to results on the integrability of $x^{-\gamma}L(1/x)f(x)$, where $L(x)$ is slowly increasing (i.e., $L(tx)/L(x) \to 1$ as $x \to \infty$). If $0 < \gamma < 2$ for a sine series, or $0 < \gamma < 1$ for a cosine series, then

$$x^{-\gamma}L(1/x)f(x) \in L$$

if and only if $\sum n^{\gamma-1}L(n)\lambda_n$ converges. If $L(x)$ is convex and nondecreasing, then if $f(x)$ is a sine series, $L(1/x)f(x) \in L$ if and only if $\sum n^{-1}L(n)\lambda_n$ converges; if $f(x)$ is a cosine series, $\sum \lambda_n$ converges, and $f(0)=0$, then $x^{-1}L(1/x)f(x) \in L$ if and only if $\sum_{n=1}^{\infty} n^{-1}L(n)\sum_{\nu=n}^{\infty} \lambda_\nu$ converges. If $f(x)$ is a cosine series, $\sum \lambda_n$ converges, $f(0)=0$, and $1 < \gamma < 3$, then $x^{-\gamma}L(1/x)f(x) \in L$ if and only if $\sum n^{\gamma-1}L(n)$ converges. The authors also prove two lemmas on slowly increasing functions: if $L(x)$ is convex, then $x\{1-L(x+a)/L(x)\} \to 0$ for each fixed a; and

$$0 < An^\alpha L(n) \leq \sum_{\nu=1}^{n} \nu^{\alpha-1}L(\nu) \leq Bn^\alpha L(n)$$

for $\alpha > 0$. *R. P. Boas, Jr.* (Evanston, Ill.).

Baiada, Emilio. Il corpo convesso di Carathéodory. Ann. Mat. Pura Appl. (4) **39** (1955), 75–85.

Let $f(x) = \frac{1}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$ have bounded variation and normalized discontinuities on $(0, 2\pi)$. Carathéodory has found [Rend. Circ. Mat. Palermo **32** (1911), 193–217] a necessary and sufficient condition in terms of the Fourier coefficients a_n and b_n that $f(x) \geqq 0$. This condition is that for each positive n, the point $(a_1, b_1; a_2, b_2; \cdots; a_n, b_n) = A_n$ in $2n$-dimensional space should be interior to the convex hull K_n of the curve given parametrically by

$$x_k = \cos k\theta,\ y_k = \sin k\theta,\ k = 1, 2, \cdots, n.$$

In the infinite-dimensional space S of sequences $A = (a_1, b_1; a_2, b_2; \cdots; a_n, b_n; \cdots)$ let K be those points for which A_n is interior to K_n for all n. Then the condition stated above that $f(x) \geqq 0$ is that $A \in K$. In the present paper the author introduces a metric into S with the purpose of characterizing the interior and frontier of K and distinguishing between the cases $f \geqq 0$ and $f > 0$. The metric used is determined as follows: Let $\alpha_k = a_k + ib_k$, and

$$D_n(t; \alpha_1, \alpha_2, \cdots, \alpha_n) = \begin{vmatrix} t & \alpha_1 & \alpha_2 & \cdots & \cdot & \alpha_n \\ \bar{\alpha}_1 & t & \alpha_1 & \cdots & \cdot & \alpha_{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \bar{\alpha}_n & \cdot & \cdot & \cdots & \bar{\alpha}_1 & t \end{vmatrix}.$$

Then the distance $d(A, A')$ is the inf of positive t such that

$$D_n(t; \alpha_1 - \alpha_1', \alpha_2 - \alpha_2', \cdots, \alpha_n - \alpha_n') > 0$$

and

$$D_n(t; \alpha_1' - \alpha_1, \alpha_2' - \alpha_2, \cdots, \alpha_n' - \alpha_n) > 0$$

for all n. With this metric the A corresponding to a continuous function f of the class considered is interior to K if and only if f is always positive; it is on the frontier of K if and only if f somewhere vanishes. *J. W. Green.*

Tomić, M. Sur les facteurs de convergence des séries de Fourier des fonctions continues. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 23–32.

The author proves some theorems on sequences of

multipliers $\{\lambda_n\}$ for Fourier series. If $\{\lambda_n\}$ is quasi-convex, a necessary and sufficient condition for it to transform every Fourier series of a continuous function into a uniformly convergent Fourier series is that $\lambda_n \log n = O(1)$. [Cf. the following review.] If f has modulus of continuity ω and $\{\lambda_n\}$ is quasi-convex, then if $\omega(1/n)\lambda_n \log n = O(1)$, $\{\lambda_n\}$ transforms the Fourier series of f into a uniformly convergent series. An application is given to the summation of Fourier series. [Cf. Karamata and Tomić, same Publ. 8 (1955), 123–138; MR 17, 479.] *R. P. Boas, Jr.*

Karamata, J. Suite de fonctionnelles linéaires et facteurs de convergence des séries de Fourier. J. Math. Pures Appl. (9) **35** (1956), 87–95.

The author proves that a sequence $\{\lambda_n\}$ of multipliers transforms every Fourier series of a continuous function into a uniformly convergent Fourier series if and only if

$$\int_0^\pi |\tfrac{1}{2}\lambda_0 + \sum_{\nu=1}^{n} \lambda_\nu \cos \nu t|\, dt = O(1).$$

He deduces the first result of Tomić reviewed above.

R. P. Boas, Jr. (Evanston, Ill.).

Mohanty, R. On the summability $|C, 1|$ of Fourier series. Bull. Calcutta Math. Soc. **47** (1955), 53–54.

A series is said to be $|C, 1|$ summable if the sequence $\{\sigma_n\}$ of the arithmetic means of its partial sums has bounded variation. It is known that $|C, 1|$ summability of a Fourier series is in general not a local property. However, the following is proved: If $\sum_1^\infty |A_n(x)|n^{-1} \log n < \infty$, where $A_n(x) = a_n \cos nx + b_n \sin nx$ is the nth term of the Fourier series of f, then the $|C, 1|$ summability of the Fourier series at the point x depends only on the behavior of f in a neighborhood of x.

W. Rudin.

Kinukawa, Masakiti. On the Cesàro summability of Fourier series. II. Tôhoku Math. J. (2) **7** (1955), 252–264.

[For part I see same J. (2) **6** (1954), 109–120; MR **16**, 1100]. Let $\varphi_\beta(t)$ be the βth integral of $\varphi(t) = f(x+t) + f(x-t) - 2s$, where $f(t)$ is a Lebesgue integrable function of period 2π. Let $\Delta^m u\varphi(t)$ be the mth difference function of $\varphi(t)$ with spacing u. Let $\Delta \geqq 1$, $\gamma \geqq \beta > 0$. The following conditions are shown to be sufficient for the (C, ϱ) summability of the Fourier series of $f(t)$ at $t = x$.

1) $0 \leqq \varrho < 1$, $\gamma = \Delta\beta - \varrho(\Delta - 1)$, $\varphi_\beta(t) = o(t^\gamma)$ as $t \to 0$

and

$$\lim_{u \to 0} u^\varrho \int_u^\pi u^{1/\Delta} |\Delta^m u\varphi(t)| t^{-1-\varrho} dt = 0.$$

2) $\varrho > -1$, $\gamma = \Delta\beta - \varrho(\beta+1)(\Delta-1)(1+\varrho)^{-1}$, $\varphi_\beta(t) = o(t^\gamma)$ as $t \to 0$

and

$$\int_0^t |d[u^\Delta \varphi(u)]| = O(t) \text{ as } t \to 0.$$

P. Civin (Eugene, Ore.).

Kanno, Kôsi. On the Cesàro summability of Fourier series. II. Tôhoku Math. J. (2) **7** (1955), 265–278.

[For part I see same J. (2) **7** (1955), 110–118; MR **17**, 361.] Let $\varphi_\beta(t)$ be the βth integral of the even integrable function $\varphi(t)$ of period 2π. The condition

(*) $\varphi_\beta(t) = o(t^\beta / \log (1/t)^{1/\gamma})$ as $t \to 0$, $\beta > 0$, $\gamma > 0$

and either of the conditions (1) and (2) are sufficient for the (C, ϱ) summability of Fourier series of $\varphi(t)$ at $t = 0$, for the value of ϱ indicated.

(1) $\int_0^t |d[t\varphi(t)/\log (1/t)^\Delta]| = O(t)$ for $0 \leqq t \leqq \eta$, $\Delta > 0$,

where $\varrho = (\Delta\gamma\beta - 1)/(1 + \Delta\gamma)^{-1}$.

(2) $\varphi(t) = O[(\log 1/t)^\Delta]$ as $t \to 0$, $\Delta > 0$,

where $\varrho = \Delta\gamma\beta/(1+\Delta\gamma)$.

P. Civin (Eugene, Ore.).

Tandori, Károly. On strong summability of Fourier series. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 457–465. (Hungarian)

Hardy and Littlewood proved [Proc. London Math. Soc. (2) **26** (1927), 273–286] that if $f(x)$ is in $L^p[0, 2\pi]$, $p > 1$, then the Fourier series of $f(x)$ is strongly summable of order r for every $r > 0$ at every Lebesgue point of order p. They gave an example which shows that this theorem is false for $p = 1$. Marcinkiewicz later proved, settling a question of Hardy and Littlewood, that if $f(x)$ is in L_1 its Fourier series is almost everywhere summable H_2, and Zygmund proved [ibid. **47** (1942), 326–350; MR **4**, 76] that its Fourier series is almost everywhere summable H_r for every $r > 0$. (The Fourier series of a function is said to be strongly summable of order r, or summable H_r at x_0 if

$$\lim (n+1)^{-1} \sum_{\nu=0}^{n} |s(f; x_0) - f(x_0)|^r = 0.)$$

The author now proves the following theorems: 1) Let $f(t)$ be in $L_1[0, 2\pi]$. Then for almost all x uniformly in K

(1) $$\lim_{h \to 0} \frac{1}{h^3 + hk} \int_{-h}^{h} |f(x+u) - f(x)|\, du \int_{u-h}^{u+k} |f(x+v) - f(x)|\, dv = 0.$$

2) If (1) is satisfied at $x = x_0$, then its Fourier series is summable H_2 at x_0.

P. Erdős.

Tandori, Károly. Über die starke Summation von Fourierreihen. Acta Sci. Math. Szeged **16**, (1955) 65–73.

German translation of the paper reviewed above.

Žak, I. E. On a generalization of a theorem of T. Wang. Stalingrad. Gos. Ped. Inst. Uč. Zap. **1953**, no. 3, 42–51. (Russian)

Let $\sigma_n^r(x)$ be the (C, r) means of the Fourier series of a periodic function $f(x)$ ($r = 0, 1, 2, \cdots$). The author proves that

$$\sum_{\nu=0}^{n} |\sigma_\nu^r(x) - f(x)| = O(n \log n)$$

at each point x, where

$$\int_0^t u^{-r-1} |\Phi_{r+1}(u)|\, du = o(t), \quad \int_0^t u^{-r} |\Phi(u)|\, du = O(t),$$

the Φ's being defined by the conditions

$$\Phi_0(t) = f(x+t) + f(x-t) - 2f(x), \quad \Phi_s(t) = \int_0^t \Phi_{s-1}(u)\, du \quad (s = 1, 2, \cdots).$$

[For $r = 0$ this includes a result obtained by Wang, Duke Math. J. **12** (1945), 77–87; MR **6**, 172.] The note contains many misprints.

A. Zygmund (Chicago, Ill.).

Ganzburg, I. M. On some methods of summation of Fourier integrals. Dnepropetrov. Gos. Univ. Nauč. Zap. **41** (1953), 145–151. (Russian)

Let $f(x) \in L(-\infty, +\infty)$, $a(t)$ and $b(t)$ be the cosine and sine Fourier transforms of f, and

$$S_\lambda(x)=\int_0^\lambda [a(t)\cos tx+b(t)\sin tx]dt$$

the partial Fourier integral of f. The main result of the paper is that if $\alpha=O(1/\lambda)$ as $\lambda\to+\infty$, then at every Lebesgue point of f we have

$$S_\lambda(x+\alpha)+S_\lambda(x-\alpha)-2\cos\lambda\alpha\cdot S_\lambda(x)-4\sin^2\tfrac{1}{2}\lambda\alpha\cdot f(x)\to 0$$

as $\lambda\to\infty$. Taking $\alpha=2\pi/3\lambda$ and $\alpha=\pi/2\lambda$ we obtain respectively

$$T_\lambda(x)\equiv\frac{1}{3}\Big[S_\lambda(x)+S_\lambda\Big(x-\frac{2\pi}{3\lambda}\Big)+S_\lambda\Big(x+\frac{2\pi}{3\lambda}\Big)\Big]\to f(x),$$

$$\frac{1}{2}\Big[S_\lambda\Big(x-\frac{\pi}{2\lambda}\Big)+S_\lambda\Big(x+\frac{\pi}{2\lambda}\Big)\Big]\to f(x).$$

The second relation had been obtained previously by I. P. Natanson [Mat. Sb. N.S. **7(49)** (1940), 313–320; MR **2**, 94]. Moreover, if $f\in L(-\infty,+\infty)$ and has a modulus of continuity $\omega(\delta)$ satisfying

$$\int_{1/\lambda}^{\infty} u^{-3}\omega(u)du=O\{\lambda^2\omega(1/\lambda)\},$$

then $f(x)-T_\lambda(x)=O(\omega(1/\lambda))$.

A. Zygmund.

Jurkat, Wolfgang; und Peyerimhoff, Alexander. Lokalisation bei absoluter Cesàro-Summierbarkeit von Potenzreihen und trigonometrischen Reihen. II. Math. Z. **64** (1956), 151–158.

If A is a linear summability method, then the F.s. (conjugate F.s.) of $KCL[0, 2\pi]$ (all functions here mentioned have period 2π) are said to have the A localization property at $\phi_0\in[0, 2\pi]$ if the F.s. (conjugate F.s.) of $f, g\in K$ are equi-A-summable at ϕ_0 whenever $f(\phi)=g(\phi)$ in a neighborhood of ϕ_0. Using their Theorem 2 and Lemmas 6 and 8 [Math. **60** (1954), 255–270; MR **16**, 351], the authors show that the F.s. $\sum_0^\infty \mathrm{Re}[a_n(f)e^{in\phi}]$ of the functions $f\in K$ have the $|C,\alpha|$, $\alpha>-1$, localization property at 0 (for all $\phi\in[0, 2\pi]$) if

$$\sum_1^\infty |\Delta\,\mathrm{Re}\,[a_n(f)/n^\alpha]|<\infty\quad\Big(\sum_1^\infty |a_n(f)|/n^\alpha<\infty\Big)$$

for each $f\in K$; the converse is true if the zero function belongs to K and if the function defined by $\lambda(\phi)f(2\phi)$ is in K whenever $f\in K$, λ being any twice continuously differentiable function satisfying $\lambda(\phi)=1-\lambda(\phi+\pi)$ for $-\pi\leqq\phi\leqq 0$ and $\lambda(\phi)=0$ for $|\phi|\leqq\pi/3$. There are corresponding results for (C,α) summability, $\alpha>-1$ (replace the inequalities above by $\mathrm{Re}\,a_n(f)=o(n^\alpha)$ and $a_n(f)=o(n^\alpha)$, respectively), and for the conjugate F.s. (replace Re by Im). They apply this and known results to obtain statements concerning Cesàro localization for the classic function spaces. For example, the F.s. and conjugate F.s. of every class K between C and L have the (C,α) localization property at every $\phi\in[0, 2\pi]$ if $\alpha\geqq 0$ and fail to have this property at each $\phi\in[0, 2\pi]$ if $-1<\alpha<0$.

A. E. Livingston (Seattle, Wash.).

See also: Džvaršeišvili, p. 962.

Integral Transforms, Operational Calculus

Delange, Hubert. Théorèmes taubériens et applications arithmétiques. Mém. Soc. Roy. Sci. Liège (4) **16** (1955), no. 1–2, 87 pp.

This memoir consists of four lectures; the first two deal essentially with the theorems called "inverse theorems of the summation process", which are examplified by Karamata in "Sur les théorèmes inverses des procédés de sommabilité" [Hermann, Paris, 1937], and which are closely related to Wiener's Tauberian theorems [Ann. of Math. (2) **33** (1932), 1–100] together with the author's work [Ann. Sci. Ecole Norm. Sup. (3) **67** (1950), 99–160, 199–242; MR **12**, 253]. In this category we find Lambert summability which the author applies to several known arithmetical problems besides the prime-number theorem.

The last two lectures are concerned with the Tauberian theorems for the Laplace-Stieltjes integral which the author has been generalizing in a series of papers [C. R. Acad. Sci. Paris **232** (1951), 465–467, 589–591, 1176–1178, 1392–1393; MR **12**, 405, 497, 605, 677]. From his generalized Tauberian theorems [Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 213–242; MR **16**, 921] are deduced several corollaries in a form susceptible of arithmetical applications. For example, let $q>1$ and r be integers, and let $\omega(n)$ denote the number of distinct prime factors in the positive integer n; then the number of numbers not exceeding x, which are "quadratfrei" and such that $\omega(n)\equiv r \pmod q$ will tend to $6\pi^{-2}x/q$ as $x\to\infty$. A particular case for $q=2$ is given in Landau, Handbuch der Primzahlen [Bd. 2, 2nd ed., Chelsea, New York, 1953, p. 606; MR **16**, 904]. Applications to many other problems are now unified by his generalized theorems. *S. Ikehara.*

Delange, Hubert. Quelques théorèmes taubériens relatifs à l'intégrale de Laplace et leurs applications arithmétiques. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 87–103.

The author gives various arithmetical applications of his generalized Tauberian theorems published in a series of papers [in particular, Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 213–242; MR **16**, 921]. The present paper is essentially a resumé of the paper reviewed above.

S. Ikehara.

Lavoine, Jean. Sur les transformées de Laplace de certaines distributions. C. R. Acad. Sci. Paris **242** (1956), 717–719.

The author states that he has prepared a table of Laplace transforms of pseudo-functions. The latter are distributions corresponding to certain non-integrable functions [L. Schwartz, Théorie des distributions, vol. I, Hermann, Paris, 1950; MR **12**, 31]. Let $f(t)$ be defined for $t>0$, not integrable over some finite interval, but of suitably restricted growth as $t\to\infty$. The Laplace transform of f is defined by

$$\mathcal{L}(f)=\text{finite part}\int_0^\infty f(t)e^{-pt}dt$$

whenever this finite part in the sense of Hadamard exists. The author lists a number of examples such as $f(t)=t^{-s-1}$, $\operatorname{cosec} t$, $J_0(t)/t$. [The theory of Laplace transforms of distribution was developed by L. Schwartz [Medd. Lunds Univ. Mat. Sem. Tome Supplémenta (1952), 196–206; MR **14**, 639]; a discussion from the point of view of applied mathematics was given by the reviewer [Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 368–378, 379–389, 483–493, 494–503, 663–674; MR **17**, 63, 354, 594]. *J. Korevaar* (Madison, Wis.).

Bhatnagar, K. P. Certain theorems on self-reciprocal functions. Bull. Calcutta Math. Soc. **47** (1955), 43–52.

In earlier papers [same Bull. **46** (1954) 179–199, 251–252; Acad. Roy. Belg. Bull. Cl. Sci. (5) **39** (1953), 42–69; MR **17**, 261; **14**, 977] the author introduced the integral transforms with kernels $\omega_{\mu,\nu}$, $\omega_{\mu,1}$,.... Here he investigates self-reciprocal functions for these transforms.

A. Erdélyi (Pasadena, Calif.).

Guinand, A. P. Matrices associated with fractional Hankel and Fourier transformations. Proc. Glasgow Math. Assoc. **2** (1956), 185–192.

The author extends the results of R. Barrucand [C. R. Acad. Sci. Paris **230** (1950), 1727–1728; MR **12**, 20] to Hankel transforms with kernels $J_\alpha(2x^{\frac{1}{2}})$, where $\mathrm{Re}\,(\alpha)>-1$, and to Fourier sine and cosine transforms. Let $f(x)$ belong to $L^2(0,\infty)$ and let $g(x)$ denote its Hankel transform $\int_0^\infty f(t)J_\alpha[2(xt)^{\frac{1}{2}}]dt$. If a_n $(n=0, 1, 2, \cdots)$ denotes a Laguerre integral transform of $f(x)$ defined by

$$n!\int_0^\infty e^{-x}x^{\alpha/2}L_n^{(\alpha)}(x)f(x)dx=\Gamma(n+\alpha+1)a_n,$$

and if b_n denotes the corresponding Laguerre transform of $g(x)$, then

$$b_n=\sum_{m=0}^{n}(-1)^m\binom{n}{m}a_m \text{ and } a_n=\sum_{m=0}^{n}(-1)^m\binom{n}{m}b_m.$$

When $\alpha=\pm\frac{1}{2}$ the above theorem reduces to a corresponding one on Fourier and Hermite transforms. The paper contains some further results of a similar nature.

R. V. Churchill (Ann Arbor, Mich.).

Stanković, B. Inversion et invariantes de la transformation généralisée de Hankel. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 37–52.

Let

$$\phi(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(\alpha k+\beta)}.$$

The author proves the inversion formula

$$G(x)=\int_0^\infty \phi(\nu,\mu+1;-x^\nu y)y^\mu g(y)dy \quad (\mu>-1,\ \nu>0),$$

$$x^{(\mu+1)(\nu-1)/\nu}g(x)=\int_0^\infty \phi\Big(\frac{1}{\nu},\frac{\mu+1}{\nu};-x^{1/\nu}y\Big)y^\mu G(y)dy$$

under conditions much less restrictive than those established by R. P. Agarwal [Ann. Soc. Sci. Bruxelles. Sér. I. **64** (1950), 164–168; MR **12**, 605]. He also investigates the general form of self-reciprocal functions for this integral transformation, showing that for self-reciprocal functions, $g(x)=G(x)$, for which

$$\gamma(s)=\int_0^\infty e^{-sx}x^\mu g(x)dx$$

exists, $\gamma(s)$ must be of the form

$$s^{-(\mu+1)/(\nu+1)}\omega\Big(\frac{\log\log s}{\log(-\nu)}\Big),$$

where $\omega(x)$ is a periodic function of period 1. For $\nu=1$ one obtains known results on Hankel transforms.

A. Erdélyi (Pasadena, Calif.).

Sarkar, G. K. On certain theorems on operational calculus and some properties of the generalised k-function of Bateman. Bull. Calcutta Math. Soc. **47** (1955), 81–86.

In the first part of this paper, the author gives the connection between the operational images of $h(x)$ and $h(2x^{\frac{1}{2}})$; in the second part, he evaluates some integrals involving confluent hypergeometric functions.

A. Erdélyi (Pasadena, Calif.).

See also: Berkeš, p. 967; Kilpi, p. 989.

Special Functions

★ **Lense, J. Elliptische Funktionen und Integrale.** Handbuch der Physik. Bd. I. Mathematische Methoden I, pp. 120–146. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. DM 72.00.

This is a condensed and very lucid account of elliptic functions and integrals. The scope of the article will be seen from the following list of titles of its sections: Doubly-periodic functions. The $\wp$-function. The ζ-function. The σ-function. Elliptic functions. Elliptic integrals. The Riemann surface for the square root of a quartic polynomial. Conformal mapping described by the $\wp$-function. Translation of the period-lattice. Rotation and stretching of the period-lattices. Weierstrass' normal form. Conformal mapping of two Riemann surfaces in Weierstrass' normal form. Primitive periods. Modular substitutions. Modular functions. Multiplication. Complex multiplication. Reduction of elliptic integrals. Legendre's normal form. Theta functions. Jacobian functions. Transformation of theta functions.

A. Erdélyi (Pasadena, Calif.).

★ **Meixner, J. Spezielle Funktionen der mathematischen Physik.** Handbuch der Physik. Bd. I. Mathematische Methoden I, pp. 147–217. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. DM 72.00.

In this article the author presents the more important properties of the hypergeometric function with its special and limiting cases (Legendre functions, confluent hypergeometric functions, Bessel functions, etc.), of Mathieu, and of spheroidal wave functions. Especially in the treatment of functions of the hypergeometric type, the methods are emphasized, and the author attempts to present the information as an organic whole rather than as a collection of more or less uncorrelated results.

A. Definitions and elementary properties of the various functions of hypergeometric type. B. Linear differential equations of the second order, Fuchsian theory, integration by Laplace integrals, asymptotic expansions. C. Functional equations. The factorization method, Truesdell's F-equation with many applications to special functions, including addition and multiplication theorems, integral relations. D. Difference equations. Expansions of special functions in series of simpler functions. E. Separation of variables in the wave equation and in Maxwell's equation. Integral relations, integral equations, addition theorems. F. Mathieu functions and spheroidal wave functions.

A. Erdélyi (Pasadena, Calif.).

Mel'nikov, I. G. On irreducibility of Legendre polynomials. Ukrain. Mat. Ž. **8** (1956), 26–33. (Russian)

It is conjectured that all $P_l(x)$, l even, and all $x^{-1}P_l(x)$, l odd, are irreducible over the rational field. Let p be an odd prime, $k=1, 2, \cdots$. J. B. Holt [Proc. London Math. Soc. (2) **11** (1912), 351–356; **12** (1913), 126–132, xxxi–xxxii] proved irreducibility for $l=2^k$, 2^k+1, p, $p\pm\alpha$, $2p\pm\alpha$, where α takes on only certain positive integral values. H. Ille [Thesis, Univ. of Berlin, 1924] showed, using ideal theory, irreducibility for $l=(p-1)p^k$,

$(p-1)p^k \pm a$, $a=1, 2, 3$, and $l=(p-1)p^k-4$. Wahab [Duke Math. J. **19** (1952), 165–176; MR **13**, 648] obtained additional results. The present author gives a completely elementary demonstration of the irreducibility of $P_l(x)$ over the rationals in the known cases of $l=p$, $p-2$, $(p-1)p^k-1$ and in the new cases $l=(p+1)p^k$, $(p+1)p^k+1$. The smallest l to which known criteria do not apply and for which irreducibility is uncertain is 206.

The author's proof is based upon the following facts. Theorem: Suppose $f(x)=\sum_0^n a_k x^{n-k}$ is a polynomial with integral coefficients where all coefficients except one (not a_0 or a_n) are divisible by p with a_0 and a_n divisible by p^α and p^β, respectively; then $f(x)$ has no more than $\alpha+\beta$ polynomial factors which are irreducible over the rational field. Lemma: Let a and b be natural numbers whose p-adic expansions are $\sum_0^s \alpha_i p^i$ and $\sum_0^t \beta_i p^i$, respectively; then

$$\sum_1^\infty \left(\left[\frac{a+b}{p^m} \right] - \left[\frac{a}{p^m} \right] - \left[\frac{b}{p^m} \right] \right) = 0$$

if and only if $\alpha_i+\beta_i<p$, $i=0, \cdots, \max(s, t)$, $[x]$ being the greatest integer in x. Lemma: Let $2a=\sum_0^s \alpha_i p^i$ be a positive even integer with p-adic expansion as shown, 2μ be the number of odd integers among the α_i's, and ϱ_k be the number of even α_i standing between the kth odd α_i and the $(k+1)$st, then

$$\sum_1^\infty \left(\left[\frac{2a}{p^m} \right] - 2\left[\frac{a}{p^m} \right] \right) = \mu + \varrho_1 + \varrho_3 + \cdots + \varrho_{2\mu-1}.$$

N. D. Kazarinoff (Ann Arbor, Mich.).

de Castro, Antonio. Recurrence formula for Legendre polynomials. Rev. Mat. Hisp.-Amer. (4) **15** (1955), 153–160. (Spanish)

The author obtains the expansion of a product of two Legendre polynomials in a series of Legendre polynomials [see also J. A. Gaunt, Philos. Trans. Roy. Soc. London. Ser. A. **228** (1929), 151–196], and applies this to a problem in nuclear physics. *A. Erdélyi* (Pasadena, Calif.).

Chatterjee, Phatik Chand. On a generalisation of Hermite's polynomial. I. Bull. Calcutta Math. Soc. **47** (1955), 27–41.

The author defines two classes of polynomials, fails to recognize that these are Laguerre polynomials in disguised form, and that his results are paraphrases of known formulas involving Laguerre polynomials.

A. Erdélyi (Pasadena, Calif.).

Berkeš, Branko. Einige Formeln über unendlichen Reihen Besselscher Funktionen. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 161–170. (Serbo-Croatian summary)

The author uses operational calculus to obtain the partial fraction expansion of the meromorphic function $x^{-\nu}J_\nu(x)$ cosec x [which is also an obvious example of E. T. Whittaker's cardinal series] and some similar expansions. *A. Erdélyi* (Pasadena, Calif.).

★**Tricomi, Francesco G. Funzioni ipergeometriche confluenti.** Edizioni Cremonese, Roma, 1954. xv+309 pp. 3500 Lire.

The present book differs considerably from the earlier Lezioni sulle funzioni ipergeometriche confluenti [Gheroni, Torino, 1952; MR **14**, 269]: it is more complete, giving a comprehensive picture of the theory and applications of confluent hypergeometric functions, and assumes more preparation on the part of the reader.

Chapter I.

$$xy''+(c-x)y'-ay=0 \tag{1}$$

is introduced as the canonical form of the confluent hypergeometric equation. Several differential equations reducible to (1) are discussed. If c is not an integer, Kummer's series,

$$\Phi(a, c; x)=\sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(c)}{n!\Gamma(a)\Gamma(c+n)} x^n, \tag{2}$$

and $x^{1-c}\Phi(a-c+1, 2-c; x)$ form a fundamental system of solutions of (1). The function $\Phi^*(a, c; x)=\Phi(a, c; x)/\Gamma(c)$ is an entire function of a, c, x. Investigation of the elementary properties of Φ including Kummer's transformation, recurrence and differentiation formulas, integrals, series, special cases. Tricomi's expansion of Φ^* in a series of Bessel functions. Chapter II. Integration of (1) by means of Laplace integrals leads to (1) and also to

$$\Psi(a, c; x)=\frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1}(1+t)^{c-a-1} dt. \tag{3}$$

Elementary properties of Ψ, connection between Ψ and Φ. Other integral representations, including Mellin-Barnes type integrals. Miscellaneous results including derivatives of confluent hypergeometric functions with respect to the parameters. Chapter III. Asymptotic expansions and zeros of confluent hypergeometric functions. Graphs of these functions for real a, c, x. Chapter IV. Special cases. Incomplete gamma functions, error functions and related functions, exponential integral and related functions, parabolic cylinder functions. Chapter V. Applications to the two-body problem of wave mechanics, bending of elastic plates, resultant of a large number of random vectors, water waves, integers which are sums of two kth powers, reflection of electromagnetic waves on a parabolic cylinder.

The book includes a table of Laplace-transform pairs involving confluent hypergeometric functions, a list of references, and an index: it is beautifully printed, and very few misprints were noticed by the reviewer.

A. Erdélyi (Pasadena, Calif.).

Frank, Evelyn. A new class of continued fraction expansions for the ratios of hypergeometric functions. Trans. Amer. Math. Soc. **81** (1956), 453–476.

The well-known linear relations between the hypergeometric functions $F(a, b, c, z)$ and any two of its contiguous functions are easily and usually written as continued fractions of Gauss:

$$\frac{F(a,b+1,c+1,z)}{F(a,b,c,z)} = \frac{1}{1} - \cfrac{\frac{a(c-b)z}{c(c+1)}}{1} - \cfrac{\frac{(b+1)(c-a+1)z}{(c+1)(c+2)}}{1} - \cdots$$

In this paper a new class of continued-fraction expansions for the ratio of two such functions is considered; these are derived for the reciprocal ratio

$$F(a, b, c, z)/F(a, b+1, c+1, z)$$

and also for other ratios where the numerators are $F(a, b, c, z)$ and the denominators are respectively $F(a+1, b, c+1, z)$, $F(a, b, c+1, z)$, $F(a+1; b+1, c+1, z)$,

$F'(a, b, c, z)$. Then the regions of convergence for these continued fractions are studied. The methods of proofs are classical (Euler, Perron). As special cases of the preceding results the author obtains new expansions for elementary functions easily deduced from hypergeometric functions: $(1+z)^{-n}$, $z/\log(1+z)$, $\tan z/z$, $\sin z/z$.

R. Campbell (Caen).

Touchard, J.; et van der Pol, Balth. Equations différentielles linéaires vérifiées par certaines fonctions modulaires elliptiques. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 166–169.

The Fourier expansions for the Eisenstein series can be expressed in the form

$$\sum_{-\infty}^{\infty}{}' \sum_{-\infty}^{\infty} (m+n\tau)^{-2k} = \alpha_{2k-1}(t) \sum_{m=-\infty}^{\infty}{}' m^{-2k},$$

where $t=-2\pi i\tau$ and

$$\alpha_{2k-1}(t) = 1+[2/\zeta(1-2k)] \sum_{n=1}^{\infty} e^{-nt}\sigma_{2k-1}(n).$$

The functions α_3 and α_5 are essentially the same as the Weierstrass invariants g_2 and g_3. The authors consider the powers

$$y_k(\tau) = [\alpha_3^3(t) - \alpha_5^2(t)]^{-k/12}$$

for $k=1, 2, \cdots, 12$. The quotient $y_k^{(k+1)}(\tau)/y(\tau)$ is a modular form of dimension $2k+2$ and can be expressed as a polynomial in α_3 and α_5 with rational coefficients. This, of course, results in a linear differential equation satisfied by y_k. The coefficients are determined explicitly for $k=1, 2, \cdots, 12$. In some instances, namely $k=3, 4, 6$ and 12, the differential equation which results is simpler when expressed in terms of the higher α's rather than in terms of α_3 and α_5. *T. M. Apostol* (Pasadena, Calif.).

Hines, Jerome. A generalization of the *S*-Stirling numbers. Math. Mag. 29 (1956), 200–203.

The S-Stirling numbers S_p^n are defined as follows:

$$(1+x)(1+2x)\cdots(1+nx) = S_0^n + S_1^n x + \cdots + S_p^n x^p + \cdots,$$

where p and n are integers. The purpose of this paper is to introduce a Stirling function generalizing the Stirling numbers for any p and n, and including them as a special case. The author proceeds to establish a recursion formula, from which can be derived the equation

$$\int_{n-1}^{n} = \Gamma(n+1) \sum_{1}^{n} p^{-1}.$$

Thus introducing the "polygamma function" of order r,

$$\psi_r(n+1) = \frac{1}{\Gamma(n+1)} D_n{}^r\Gamma(n+1) = \frac{1}{\Gamma(n+1)} \int_0^{\infty} e^{-x} x^n (\log x)^r dx,$$

and the function $\xi_r(n) = \psi_r(n) - \psi_r(1)$ (where $\psi_r(1) = -C$, Euler's constant), he proves that

$$S_p^n = \frac{\Gamma(n+1)}{\Gamma(n-p+1)} \xi_{n-p}(n+1),$$

which realizes the interpolation. *R. Campbell.*

Kazarinoff, Nicholas D. A note on two generating functions for Legendre functions. Proc. Amer. Math. Soc. 7 (1956), 230–231.

The author sums

$$\sum_{n=0}^{\infty} P_{m+n}^{m}(z) \frac{t^{m+n}}{(m+n)!} \text{ and } \sum_{n=0}^{\infty} P_{m+n}^{m}(z) \frac{t^{m+n}}{(2m+n)!},$$

thus correcting two errors in C. Truesdell's "A unified theory of special functions" [Princeton, 1948, p. 101 equation (58) and p. 105 equation (66); MR 9, 431] and an error in A. Erdélyi et al., "Higher transcendental functions" [vol. III, McGraw-Hill, New York, 1955, p. 266 equation (17); MR 16, 586]. *A. Erdélyi.*

Kestin, Joseph; and Persen, Leif N. On the error function of a complex argument. Z. Angew. Math. Phys. 7 (1956), 33–40.

The authors set

$$K(z) = \frac{2z}{\sqrt{\pi}} \int_0^z e^{z^2-t^2} dt$$

and express the real and imaginary parts of $K(re^{i\phi})$ in terms of elementary functions and the integrals

$$\int_0^{\infty} \frac{u^{n-1/2} e^{-ur} du}{u^2+2u\cos\phi+1} \quad (n=1, 2).$$

A. Erdélyi (Pasadena, Calif.).

van der Blij, F. The value of a certain Epstein zeta function. Nieuw Arch. Wisk. (3) 4 (1956), 13–14.

Let $r_k(n)$ denote the number of representations of n as a sum of k squares of integers, and define the entire function $\xi_k^*(s)$ by the Dirichlet series

$$\xi_k^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} r_k(n).$$

The author proves that

$$\xi_6^*(3) = -\frac{\pi^3}{2}\log 2 + \frac{3\pi}{4}\xi(3),$$

where $\xi(s)$ is Riemann's zeta function.

A. Erdélyi (Pasadena, Calif.).

See also: Carlitz, p. 944; Bhatnagar, p. 966; Guinand, p. 966; Stanković, p. 966; Sarkar, p. 966; de Castro, p. 969; Miller, p. 1012; Sharp, Gove and Paul, p. 1013.

Ordinary Differential Equations

★**Erdélyi, A. Differential equations with transition points. I. The first approximation.** Tech. Rep. 6. Department of Mathematics, California Institute of Technology, Pasadena, 1955. 22 pp.

The behavior of the solutions of the differential equation $d^2y/dx^2 + [\lambda^2 p(x) + r(x, \lambda)]y = 0$ is investigated for large $|\lambda|$ and for $a<x<b$ under the central hypothesis that $p(x)$ have a single simple zero on (a, b). Only dominant terms of the solution forms are obtained in the present report. This report is essentially, therefore, a refinement of well-known work of R. E. Langer [Trans. Amer. Math. Soc. 33 (1931), 23–64] and others. The distinguishing feature of the present investigation is that a and/or b are permitted to be singular points of the differential equation. The interval (a, b) may be unbounded. Langer [loc. cit.] and others have allowed singular points at $x=\pm\infty$ under assumptions similar to those below. If one adopts the notations

$$\varphi(\varphi')^2 = p(x), \quad \psi(x) = (\varphi')^{\frac{1}{2}}(1+|\lambda|^{\frac{1}{2}}|\varphi(x)|^{\frac{1}{2}}),$$
$$F(x, \lambda) = (\varphi'''/\varphi') - [3(\varphi''/\varphi')^2/2] - r(x, \lambda),$$

the principal assumptions upon which the results are

based are: $\varphi \in C^3$, $\varphi'>0$ on (a, b); for allowed λ, $F \in C$ on (a, b); and $|G(t, \lambda)| \leq \Lambda(\lambda)g(t)$, where $G(t, \lambda)=\lambda^{\frac{1}{2}}F(t, \lambda)/\varphi^{\frac{3}{2}}(t)$, $\Lambda(\lambda)/\lambda$ is bounded in λ for allowed λ, and $g(t)$ is integrable over (a, b). *N. D. Kazarinoff* (Ann Arbor, Mich.).

Barnett, I. A. Particular integrals of linear differential equations. Amer. Math. Monthly **63** (1956), 245–246.

Voronovskaia, E. V. On a modification of Čaplygin's method for a differential equation of first order. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **10** (1956), no. 1(16), 36–43. (Romanian)

Romanian version of a Russian article [Prikl. Mat. Meh. **19** (1955), 121–126; MR **16**, 962].

Zaidman, S. Evaluations de la distance entre les zéros des solutions des équations différentielles. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6-7, 47–54. (Romanian. Russian and French summaries).

Using lemmata due to de la Vallée-Poussin and Beurling, the following theorems are proven: Theorem 1. Consider the differential equation

(1) $y^{(n)}+A_2(x)y^{(n-2)}+\cdots+A_n(x)y=0$, $n\geq 2$,

with $A_\nu(x)$ $(\nu=2, 3, \cdots, n)$ continuous on the interval $[a, b]$, $|A_\nu(x)|\leq L_\nu$. Let h_1 be the positive root of (2) $\sum_{\nu=2}^{n} L_\nu h^\nu/(\nu-1)!=4$. Then a unique solution of (1) passes through the n points (a_ν, α_ν), $a\leq a_1<\cdots<a_n\leq b$, provided that $a_n-a_1\leq h_1$. Theorem 2. Consider the differential equation (3) $y^{(n)}+F(x, y, y', \cdots, y^{(n-2)})=0$, where F is continuous in all arguments and satisfies a Lipschitz condition inside the closed region $R(\alpha\leq x\leq\beta, c_i\leq y^{(i)}\leq d_i)$ $(i=0, 1, \cdots, n-2)$, and $F(x, 0, 0, \cdots, 0)=0$. Two distinct solutions of (3) belonging to R cannot have more than $n-1$ points in common, if the distance between the extreme abscissae is less than the positive root h_1 of (2). Theorem 3. If in (4) $y''=F(x, y)$, F is continuous in $D(a\leq x\leq b, c\leq y\leq d)$ and satisfies in D the Lipschitz condition $|F(x, y_1)-F(x, y_2)|<L(y_1-y_2)$ and $F(x, 0)=0$, then any two consecutive zeros a, b of a solution of (4) in D, satisfy $b-a>2\cdot 3^{\frac{1}{2}}L^{-\frac{1}{2}}$. *E. Grosswald.*

Sansone, G. Linee caratteristiche delle equazioni differenziali ordinarie omogenee. Matematiche, Catania **10** (1955), 18–19.

Auszug aus einer Vorlesung über Charakteristiken des Systems $dx/dt=X(x, y)$, $dy/dt=Y(x, y)$, wo $X(x, y)$ und $Y(x, y)$ homogene Polynome mit der Eigenschaft $X(0, 0)=Y(0, 0)=0$, $X^2+Y^2\neq 0$ für $x^2+y^2\neq 0$ sind. Es werden bekannte Resultate erwähnt und einige davon vertieft. *M. Zlámal* (Brno).

de Castro, Antonio. A method of integrating differential equations reducible to the form of Bessel's equation. Gac. Mat., Madrid (1) **7** (1955), 167–173. (Spanish)

Given an ordinary linear differential equation of the second order, the author shows how to decide whether or not the general solution is of the form $v(x)Z_p(u(x))$, where $Z_p(z)$ is the general solution of Bessel's equation. *A. Erdélyi* (Pasadena, Calif.).

★**Mandel'štam, L. I. Lekcii po kolebaniyam (1930-1932 gg.). [Lectures on vibrations (1930-1932).]** Izdat. Akad. Nauk SSSR, Moscow, 1955. 503 pp. 23 rubles.

Mandel'stam's lectures at the Moscow State University for 1930–31 and 1931–32 as worked out by G. S. Gorelik and S. M. Rytov.

Colombo, Giuseppe. Oscillazioni persistenti di un sistema non lineare dissipativo dovute al ritardo della forza di richiamo. Ann. Univ. Ferrara. Sez. VII. (N.S.) **4** (1954–1955), 33–50.

The differential-difference equation

$$\ddot{x}+\varepsilon D\dot{x}+x_*-\beta x_*^3=0 \quad [\varepsilon>0,\ x_*=x(t-\varepsilon)]$$

occurs in the theory of nonlinear control systems having retarded restoring forces. In this paper the solutions of the equation are studied by the methods of phase plane analysis and perturbation theory. It is shown that if $0.7<D<1$ and $\beta>0$, and if ε is sufficiently small, the equation has stable oscillatory solutions in the neighborhoods of certain of the periodic solutions of the equation $\ddot{x}+x-\beta x^3=0$. The author conjectures, on the basis of physical considerations, that if $\beta<0$, the solutions are completely unstable. *L. A. MacColl* (New York, N.Y.).

Colombo, Giuseppe. Moti di regime di un sistema nonlineare autonomo in due gradi di libertà, con debole accoppiamento capacitivo. Rend. Sem. Mat. Univ. Padova **24** (1955), 400–420.

The solutions of the system of equations

$$\text{(A)}\qquad \begin{aligned} &\ddot{x}+f_1(x, \dot{x})+g_1(x)=\varepsilon F_1(x, \dot{x}, y, \dot{y}) \\ &\ddot{y}+f_2(y, \dot{y})+g_2(y)=\varepsilon F_2(x, \dot{x}, y, \dot{y}) \end{aligned}$$

are studied under the assumption that the functions f_i, g_i, F_i are of class C^3 and ε is a small parameter. It is assumed also that the equations

$$\ddot{x}+f_1(x, \dot{x})+g_1(x)=0,\ \ddot{y}+f_2(y, \dot{y})+g_2(y)=0$$

have periodic solutions, represented by stable limit cycles in the respective phase planes. It is found that if ε is sufficiently small, there exists a surface V^*, in the $x\dot{x}y\dot{y}$-phase space, which is topologically a torus, and which is covered simply by ∞^1 of the trajectories defined by (A). The trajectories on V^* are stable, in the sense that the trajectory passing through any point of a certain neighborhood of V^* approaches a trajectory on V^* asymptotically as $t\to+\infty$. *L. A. MacColl.*

Myškis, A. D.; and Panovko, Ya. G. Action of a disturbing force of variable frequency and amplitude on a linear system with one degree of freedom. Inžen. Sb. **22** (1955), 33–41. (Russian)

The paper considers the behavior of the function $\int_0^t P(s)\sin p(t-s)ds$, under various assumptions concerning the forcing function $P(s)$. *R. Bellman.*

Conti, Roberto. Sulla stabilità dei sistemi di equazioni differenziali lineari. Riv. Mat. Univ. Parma **6** (1955), 3–35.

Wie bekannt, ist der lineare homogene System (1) $\dot{y}=A(t)y$ stabil im Sinne von Lyapunov gerade dann, wenn (2) $\|Y(t)\|\leq c_1$ für $t\geq 0$ gilt, wo $Y(t)$ die Fundamentalmatrix ist. Wenn $\|Y(t)Y^{-1}(\tau)\|\leq c_2$ für $0\leq\tau\leq t$ gilt, so ist (1) gleichmässig stabil im Sinne von Lyapunov. Wenn ausser (2) noch $\|Y^{-1}\|\leq c_3$ für $t\geq 0$ gilt, so nennt der Autor den System (1) stabil im strengen Sinne. Der Autor beweist, dass einige bekannte, die Stabilität von (1) garantierende Kriterien in Wirklichkeit die gleichmässige Stabilität garantieren [z.B. die von Cesari, Ann. Scuola Norm. Sup. Pisa (2) **9** (1940), 163–186; MR **3**, 41]. Weiter führt er eine Reihe von Kriterien für die Stabilität im strengen Sinne und ein Kriterium für die gleichmässige Stabilität der nichthomogenen Systeme ein. *M. Zlámal* (Brno).

★ **Young, Dana; and Hess, P. N. On the stability of harmonic solutions of a modified form of Duffing's equation.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 79–84. American Society of Mechanical Engineers, New York, 1955. $9.00.

For a certain easily calculated pair of values of the constants F_1, F_2 the function $A\cos\omega t$ is a solution of the equation

$$\ddot{x}+x+\beta x^3=F_1\cos\omega t+F_2\cos 3\omega t.$$

In this paper the first-order stability of this solution is studied. Since the relevant variational equation is the Mathieu equation, a full discussion is possible. The theoretical predictions were checked experimentally by means of an analogue computer, with generally satisfactory results. The computer was also used to determine the nature of the solutions, initially in the neighborhood of $A\cos\omega t$, under various conditions which make the latter solution slightly unstable. The experimental results obtained in this part of the study are interesting and suggestive, but the authors' tentative explanations of them seem to be inadequate. *L. A. MacColl.*

Clauser, Francis H. The behavior of nonlinear systems. J. Aero. Sci. 23 (1956), 411–434.

Expository paper.

Manaresi, Gabriella. Ulteriori limitazioni per l'ampiezza di oscillazioni non-lineari. Boll. Un. Mat. Ital. (3) 10 (1955), 537–540.

The results of a previous paper by the author [Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) 2 (1954–55), 184–189] concerning the periodic solution $x_p(t)$ of the differential equation $\ddot{x}+f(x)\dot{x}+\omega^2x=0$ are extended to the case of nonsymmetric $f(x)$. Let $f(x)<0$ for $-\delta_1<x<\delta_2$ ($\delta_1>0$, $\delta_2>0$) and $f(x)=\beta$ for $x<-\delta_1$, $f(x)=\alpha$ for $x>\delta_2$, where α, β are constants greater than ω. Then the function $F(x)=\int_0^x f(x)dx$ vanishes at two points $x=-h_1$, $x=h_2$. Let $M=\max(F(-\delta_1), |F(\delta_2)|)$. The author constructs a certain region R in the plane of the variables x, $y=\dot{x}+F(x)$ such that $x_p(t)$ remains in R for all t. This enables her to derive the inequalities

$$-\frac{2\omega\alpha(h_1+h_2)}{\beta(\alpha-\omega)}-\frac{M}{\beta}-h_1\leq x_p(t)\leq\frac{2\omega\beta(h_1+h_2)}{\alpha(\beta-\omega)}+\frac{M}{\alpha}+h_2.$$

If also $h_1<h_2$ and $\alpha>\omega\sqrt{2}$ these bounds can be replaced by

$$-\frac{\alpha}{\beta}\frac{2\sqrt{(2)}\omega h_2}{\alpha-\omega\sqrt{2}}-\frac{M}{\beta}-h_1\leq x_p(t)\leq\frac{3\sqrt{(2)}\omega h_2}{\alpha-\omega\sqrt{2}}+\frac{M}{\alpha}+h_2.$$

W. R. Wasow (Los Angeles, Calif.).

Colombo, Giuseppe. Sopra il fenomeno dell'azione asincrona. Rend. Sem. Mat. Univ. Padova 24 (1955), 353–395.

The asynchronous action mentioned is the phenomenon, observable in some nonlinear systems, in which a periodic driving force, with period T_0, maintains a motion which is approximately periodic with a period $T_1>T_0$. This action is distinguished from the more familiar subharmonic resonance by the fact that the relation between T_1 and T_0 is not simple, and may be arbitrary within certain limits. Examples of asynchronous action have been discussed approximately by Hartley [Bell System Tech. J. 15 (1936), 424–440] and by Béthenod [C. R. Acad. Sci. Paris 207 (1938), 847–849]. In the present paper the author describes two rather complicated mechanical systems which exhibit asynchronous action in simple forms admitting of a full mathematical treatment. *L. A. MacColl* (New York, N.Y.).

Corduneanu, C. Problèmes aux limites sur un demi-axe pour les équations différentielles non-linéaires du second ordre. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. 6 (1955), 163–171. (Romanian. Russian and French summaries)

Let D be the half-plane $x\geq 0$, $-\infty<y<\infty$. If $f(x, y)$ is continuous, f_y exists, $0<m\leq f_y\leq M$ in D, and $f(x, 0)$ is bounded for $x\geq 0$; then the differential equation $y''=f(x, y)$ has a bounded solution on $x\geq 0$ satisfying any given initial condition of the usual kind at $x=0$. A second theorem states a further result under the additional assumptions that $f_y(x, y)$ is continuous, and $f(x, y)$ is asymptotically almost periodic in x, uniformly in y in some strip. *A. Erdélyi* (Pasadena, Calif.).

★ **Minc, R. M. Investigation of the trajectories at infinity of three differential equations.** Pamyati Aleksandra Aleksandroviča Andronova [In memory of Aleksandr Aleksandrovič Andronov], pp. 499–534. Izdat. Akad. Nauk SSSR, Moscow, 1955. 36.40 rubles.

The author considers the qualitative behavior of the trajectories of the real differential system

1) $$\dot{x}=P(x, y, z),\ \dot{y}=Q(x, y, z),\ \dot{z}=R(x, y, z),$$

where P, Q, R are polynomials in Euclidean 3-space E^3. For the behavior near each of the isolated criticial points, it is assumed that the system is "insensitive" or structurally stable. This means that the characteristic roots have nonzero real parts. Such critical points are classified as 1) node, 2) focus, 3) saddle, and 4) saddle-focus depending on whether 1) all roots real and of the same sign, 2) exactly one real root but all real parts of the same sign, 3) all roots real but not all of the same sign, 4) exactly one real root but real parts of roots not all of same sign, respectively.

To determine the global behavior of the solutions of 1) the author compactifies E^3 to the projective space P^3. Thus the system 1) is first transformed to an equivalent system in the unit ball B^3. Then, after a change of time scale near infinity, that is, near the surface S^2 of B^3, one obtains a differential system in the closed ball which is the same at antipodal points of the surface S^2.

The surface S^2 is the union of solution curves and one applies the Poincaré-Bendixson theory on S^2 to infer the behavior of the solution curves in the whole ball. For example, critical points or closed orbits on S^2 can be in the limit set of some solution lying in the interior of B^3. Also trajectories on S^2 whose limit sets consist of closed orbits (or closed contours), spatial saddle, or saddle-focus critical points can be contained in the limit sets of solutions lying interior to B^3. But a trajectory on S^2 one of whose limit points is a spatial node or focus cannot be contained in the limit set of a solution interior to B^3.

The author works out a number of examples, including all the linear equations, and draws the appropriate figures, to illustrate all these possibilities.

The author calls a system 1) absolutely unstable at infinity if no trajectory interior to B^3 has ω-limit points

on the surface S^2. For certain systems of the form 1) the author gives necessary and sufficient conditions that the system should be absolutely unstable at infinity.

L. Markus (Princeton, N.J.).

Urabe, Minoru. A supplement to "Infinitesimal deformation of the periodic solution of the second kind and its application to the equation of a pendulum". J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 113.

See same J. Ser. A. **18** (1954), 183–219; MR **17**, 264.

Kay, I.; and Moses, H. E. The determination of the scattering potential from the spectral measure function. III. Calculation of the scattering potential from the scattering operator for the one-dimensional Schrödinger equation. Nuovo Cimento (10) **3** (1956), 276–304.

Reviewed earlier as a report [Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. no. **CX–20** (1955); MR **17**, 489].

Leko, Toma. Über die Integration der Differentialgleichung $yy''+f(x)y^2=\varphi(x)$. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 171–174. (Serbo-Croatian summary)

The author determines certain cases in which the differential equation of the title can be transformed into equations which are integrable by quadratures.

L. A. MacColl (New York, N.Y.).

★**Gol'dfarb, L. S. A method of investigating nonlinear control systems based upon the principle of harmonic balance.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 177–192. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

It is the author's aim to present a method for the analysis of automatic control systems that involve a nonlinear element and whose linear part is described by differential equations of arbitrarily high order. This method establishes conditions of stability for nonlinear systems; it yields the possible oscillations within the system, their frequency, amplitude and degree of stability. The method is based on the principle of the harmonic balance and on the concept of the equivalent complex coefficient of amplification of the nonlinear element. It is essentially a modification of the method of a small parameter. *H. P. Thielman* (Ames, Ia.).

★**Smirnova, I. M. On stability of approximately determined periodic regimes of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 193–203. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

The Krylov-Bogolyubov method of harmonic balance is applied to automatic control systems in which the nonlinear part possesses an odd characteristic. The author first derives the characteristic equation which gives approximations for the regions of stability of periodic motions under the assumptions that the given system differs slightly from the linear system with a small oscillatory degree of stability. Next conditions of stability for forced oscillation are derived which are similar to those derived by Gol'dfarb for auto-oscillations. [Avtomat. i Telemeh. **8** (1947), 349–383; MR **12**, 413].

H. P. Thielman (Ames, Ia.).

★**Nadžafov, È. M. Approximate determination of periodic solutions in systems of automatic control having several nonlinearities.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 204–218. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

Several particular examples of systems which contain nonlinear elements with single-valued characteristics are considered. The amplitude and frequency of possible periodic solutions are determined by a graphical method which consists in finding the points of intersections of curves with straight lines. Nonperiodic solutions or stability conditions of solutions are not considered in this paper. *H. P. Thielman* (Ames, Ia.).

★**Cypkin, Ya. Z. On the theory of relay systems of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 329–362. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

This is a presentation of the frequency method for the study of auto-oscillations and of forced oscillations in relay systems. The method is applicable under those conditions when the linear part of the system is quite complicated and contains an inner feed back, elements with time lag and distributed parameters, that is, when the analytic method becomes useless for practical purposes, due to the computational difficulties in determining the poles of the transfer function. In applying the proposed method one starts with the construction of the amplitude-phase characteristic of the linear part of the system. This is accomplished either on the basis of the equations of the individual elements or on the basis of experimental data. By means of this amplitude-phase characteristic of the linear part one finds: 1) the stability of the position of equilibrium of the relay system; 2) the characteristic of the relay system, which determines the conditions for auto-oscillations and their period and phase shift; 3) the hodograph of the characteristic equation, which solves the problem of the stability of auto-oscillations and of forced oscillations; 4) the nature of the forced oscillations. *H. P. Thielman* (Ames, Ia.).

★**Dolgolenko, Yu. V. Sliding regimes in relay systems of indirect control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 421–438. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

The paper contains a detailed discussion of the stability of systems of the form $dx_i/dt=\sum_j a_{ij}x_j+y$, $dy/dt=f(z)$, where z is a linear form in the x_i and y. *R. Bellman.*

★**Myasnikov, N. N. On the influence of retardation on stability of automatic control.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 587–601. Izdat. Akad. Nauk SSSR. Moscow-Leningrad, 1955. (Russian)

The process under consideration leads to a characteristic equation of the form $q^3+aq^2+bq+c-e^{-q}=0$. The

author considers the influence of the time lag s upon the stability of the process under various assumptions concerning the domain of the parameters. *R. Bellman.*

Ovseevič, I. A. Determination of the upper bound of the time function of a linear system for a given frequency characteristic. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 2, 59–68. (Russian)

If $P(x)$ is the characteristic polynomial of a linear equation, $1/P(i\omega)$ is the frequency characteristic of the corresponding system, while the response to a unit input is called the time function. The derivative of the time function and the frequency characteristic are mutually connected by a Fourier transformation. Hence, by means of the Schwarz inequality, upper bounds are obtained in terms of the difference of the respective frequency characteristics, for the absolute and mean-square differences of the time functions of two systems, and for the difference of their derivatives. Mistakes are pointed out in an earlier paper by H. G. Baerwald [Phil. Mag. (7) **21** (1936), 833–869]. *S. K. Zaremba* (Wolverhampton).

Haplanov, M. G. Linear differential equations of infinite order with analytic coefficients. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1162–1165. (Russian)

A classification is made of infinite matrices that transform one Köthe-Toeplitz coordinate space E into another, E_1, based on the concept of linear dependence of rows and columns: the columns of matrix M are dependent if the homogeneous system of equations $Mc=0$ has a non-null solution $c \in E$, and the rows if the transformed system $vM=0$ has a non-null solution $v \in E_1^*$. There are four cases, each giving rise to two possibilities. For example, if the rows and columns of M are both independent then M transforms E in 1-1 manner either (i) onto E_1 (so that M has a unique inverse), or (ii) onto H, a dense proper subset of E_1 (so that an inverse matrix does not exist).

The above classification is used to study the differential equation of infinite order (1) $L[y]\equiv\sum_{k=0}^{\infty} a_k(x)y^{(k)}=f(x)$, where f, $\{a_k\}$ are analytic about $x=0$: $a_k(x)=\sum_{n=0}^{\infty} a_{nk}x^n$, $f(x)=\sum_0^{\infty} b_n x^n$, for operators L having (any) one of the properties: I. If $y \in E_\sigma$ (the class of entire functions of not exceeding first order, type σ) then $L[y]$ is analytic in $|x|<R$. II. If $y \in E_\sigma$, then $L[y] \in E_{\sigma_1}$. III. If y is analytic in $|x|<R$, then $L[y]$ is analytic in $|x|<R$. It is further supposed that the series for $L[y]$ converges uniformly about $x=0$. A number of theorems are stated concerning (1) and associated matrices, of which we quote two: 1. $L[y]$ is of class I if and only if the operator matrix

$$N=\begin{pmatrix} a_{00} & a_{01} & \cdots \\ a_{10} & a_{11} & \cdots \\ \cdot & \cdot & \cdots \end{pmatrix}$$

transforms space $A_{1/\sigma}$ into A_R. [Definition of these spaces appears in an earlier work of the author, same Dokl. (N.S.) **79** (1951), 929–932; MR **13**, 252.] 2. Let $L[y]$ be of class I. If $f(x) \in E_\sigma$, then equation (1) has a solution $y \in E_\sigma$ if and only if the vectors c, b satisfy the system $Mc=b$. {Here $M=(a_{jn}')$, where $a_{jn}'=\sum_{s=0}(1/s!)a_{j-s,n-s}$; $c=(c_0, c_1, \cdots)$, where $y(x)=\sum_0^{\infty}(c_n/n!)x^n$; and $b=(b_0, b_1, \cdots)$.} An interesting application is made to the equation $\sum_{n=0}^{\infty}(a_n+b_nx)y^{(n)}=f(x)$. *I. M. Sheffer.*

See also: Šidlovskiĭ, p. 947; Popova, p. 956; Potapov, p. 958; Touchard et van der Pol, p. 968; Haplanov, p. 972.

Partial Differential Equations

Teodorescu, N. On the necessity of the algebraic or algebraico-logarithmic character of the singularity of elementary solutions. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 939–951. (Romanian. Russian and English summaries)

"It is shown that any elementary solution of a linear partial differential equation of hyperbolic or elliptic type, with holomorphic coefficients, having the form $f(\Gamma)U+g(\Gamma)V$, where $U(x^1, \cdots, x^n, \Gamma)$ and $V(x^1, \cdots, x^n, \Gamma)$ are holomorphic functions on the characteristic conoid $\Gamma=0$, possesses on this conoid an algebraic or algebraico-logarithmic singularity···. The only epicentral waves··· are of the following three types: 1. holomorphic epicentral waves for $n=2m+1$ or $n=2m$; 2. singular epicentral waves (elementary solutions) with algebraic singularities for $n=2m+1$; 3. singular epicentral waves (elementary solutions) with algebraico-logarithmic singularities, for $n=2m$." (From the author's abstract.) *A. Erdélyi.*

Cimmino, G. Proprietà di media nella teoria delle equazioni lineari alle derivate parziali. Matematiche, Catania **10** (1955), 100–103.

Summary of expository address. *J. W. Green.*

Rašajski, B. Interprétation géométrique des fonctions caractéristiques des intégrales des caractéristiques d'une équation aux dérivées partielles à deux variables. Glas Srpske Akad. Nauka **206.** Od. Prirod.-Mat. Nauka (N.S.) **5** (1953), 3–16. (Serbo-Croatian. French summary)

Verfasser bespricht die Entwicklung der geometrischen Deutungen der Cauchyschen Charakteristikentheorie an Hand der Arbeiten von Monge, Lemonier, A. Mayer, N. Saltykow, P. du Bois Reymond und H. Léauté. Bereits 1876 diskutiert H. Léauté auf analytischem Weg das Anfangswertproblem der allgemeinen partiellen Differentialgleichung erster Ordnung und Cauchy's Formel

$$I=I_0\exp\Big[-\int\frac{z}{P}\,dx\Big].$$

Diese besagt bekanntlich, daß der Ausdruck $I=\partial z/\partial v-p\partial x/\partial v-q\partial y/\partial v$ identisch verschwindet, wenn $I_0=0$ (dasselbe gilt für den Ausdruck $\partial z/\partial v-p\partial x/\partial u-q\partial y/\partial u$, in diesem Falle als unmittelbare Konsequenz der Charakteristikengleichungen). Demgegenüber bevorzugt Verfasser geometrische Methoden und gewinnt notwendige und hinreichende Bedingungen, um aus dem allgemeinen Integral des charakteristischen Systems Lagrange's vollständigen Integrale zu erhalten. Unter den (unendlichvielen) vollständigen Integralen zeichnet Verfasser weiterhin zwei Arten näher aus. Von speziellen Cauchyschen Problemen werden Integralflächen behandelt, welche durch die Kurve

$$x=x_0,\ z=\Phi(y)$$

gehen, von allgemeineren solche, deren vorgegebene Anfangskurve durch die Gleichungen

$$x=\Phi_1(y),\ z=\Phi_2(y)$$

bestimmt ist. *M. Pinl* (Köln).

Saltykow, N. Théories analytiques et géométriques des équations aux dérivées partielles du premier ordre. Glas Srpske Akad. Nauka **206.** Od. Prirod.-Mat. Nauka (N.S.) **5** (1953), 1–2. (Serbo-Croation. French summary).

Verfasser gibt eine Übersicht über bemerkenswerte Ab-

schnitte in der Entwicklung der Integrationstheorie partieller Differentialgleichungen erster Ordnung in einer unbekannten Funktion. Sie betrifft hinsichtlich analytischer Gesichtspunkte vornehmlich die Rolle der sogenannten charakteristischen Funktionen, hinsichtlich geometrischer Methoden im Sinne von G. Monge, neue von M. B. Rašajski gewonnene Resultate. Wie zu erwarten ist, gelingt die Berechnung vollständiger Integrale aus der Kenntnis der Charakteristiken. Die Integration totaler Differentialgleichungen ist neuerdings von J. S. Aržanyh behandelt worden [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 3(55), 99–104; MR **15**, 317], dessen Resultate, wie C. Orloff bemerkt hat, auch aus älteren Ergebnissen des Verfassers (N. Saltykow) erschlossen werden können. Es handelt sich dabei zunächst um die folgenden Arbeiten des Verfassers: C. R. Acad. Sci. Paris **128** (1899), 225–227, 274–276; Soobšč. Har'kov. Mat. Obšč. (2) **6** (1899), 225–234; Über die Integration der partiellen Differentialgleichungen erster Ordnung mit einer unbekannten Funktion, Kharkow, 1899. Die in Rede stehenden Probleme wurden von Verfasser später vielfach in verallgemeinerter Form präzisiert und ihre Lösungen diskutiert [Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 4^0 (2) **6** (1925), no. 4; Méthodes classiques d'intégration des équations aux dérivées partielles du premier ordre, Mémor. Sci. Math., no. 50, Gauthier-Villars, Paris, 1931; Méthodes modernes d'intégration des équations aux dérivées partielles du premier ordre à une function inconnue, ibid., 1935); Methoden zur Integration der partiellen Differentialgleichungen erster Ordnung in einer unbekannten Funktion, Srpska Akad. Nauka, Beograd, 1947; MR **10**, 253]. — Der Generalnenner aller dieser zahlreichen Untersuchungen ist unverrückbar Cauchy's geniale Charakteristikenmethode.

M. Pinl (Köln).

Saltykow, N. Domaine d'existence d'intégrales des équations aux dérivées partielles d'ordres supérieurs au premier. Glas Srpske Akad. Nauka **206**. Od. Prirod.-Mat. Nauka (N.S.) **5** (1953), 31–38. (Serbo-Croatian. French summary)

Nach E. Goursat kann man das Existenzproblem für eine Lösung der partiellen Differentialgleichung zweiter Ordnung $r=H(x, y, z, p, q, t, s)$ in einer abhängigen z und zwei unabhängigen x, y Veränderlichen auf die Untersuchung des Systems erster Ordnung

$$(*)\qquad \frac{\partial z}{\partial x}=p,\frac{\partial p}{\partial x}=H\Big(x, y, z, p, q,\frac{\partial p}{\partial y},\frac{\partial q}{\partial y}\Big),\ \frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}$$

für die drei unbekannten Funktionen $z(x, y)$, $p(x, y)$, $q(x, y)$ zurückführen. Verfasser betrachtet den Fall der linearen partiellen Differentialgleichung

$$(**)\qquad r=As+Bt+Cp+Dq+Ez+F,$$

deren Koeffizienten von den unabhängigen Veränderlichen x, y abhängen. Dann entspricht dem System (**) das System

$$\frac{\partial Z}{\partial x}=\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial x}=$$

$$M\Big[\Big(1-\frac{x}{a}\Big)\Big(1-\frac{y}{b}\Big)\Big]^{-1}\Big(\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial y}+P+Q+Z+1\Big).$$

M bedeutet eine obere Schranke für die Beträge der Koeffizienten A, B, C, D, E, F; a und b sind Konvergenzradien der Koeffizienten A, B, C, D, E, F als Funktionen von x bzw. y. Die Anfangswerte der $x, y, Z, P, Q, \partial P/\partial y, \partial Q/\partial y$ werden zweckmäßig gleich Null gesetzt. Da dann $Z=P=Q$, ergibt sich für Z bzw. $3Z+1=\varepsilon^{-(3/2)\mathfrak{N}}U$ nach einigen Umformungen

$$Z=\frac{1}{3}(-1+\varepsilon^{\mathfrak{N}}),$$

wo $$\mathfrak{N}=\frac{3}{2}\Big\{b-y-b\Big[\Big(1-\frac{y}{b}\Big)^2+\frac{2aM}{b}\log\Big(1-\frac{x}{a}\Big)\Big]^{\frac{1}{2}}\Big\}.$$

Analog zur Theorie der gewöhnlichen linearen Differentialgleichungen folgt: die lineare partielle Differentialgleichung (**) gestattet in einem abgeschlossenen Intervall, in welchem ihre Koeffizienten reguläre Funktionen der unabhängigen Veränderlichen sind, ein reguläres Integral. Mit analogen Methoden werden weiterhin noch der Fall der allgemeinen partiellen Differentialgleichung zweiter Ordnung sowie der partiellen Differentialsysteme erster Ordnung für m unbekannte Funktionen $z_1, z_2, \cdots, z_m$ untersucht.

M. Pinl (Köln).

Beckert, Herbert. Abschätzungen bei linearen elliptischen Systemen 1. Ordnung mit zwei unabhängigen Variablen. Math. Nachr. **13** (1955), 327–342.

This paper, in part, is an application of the author's previous researches [Math. Nachr. **5** (1951), 173–208; **12** (1954), 257–272; MR **13**, 748; **17**, 741] into elliptic, linear, first-order partial differential equations in two independent variables in a bounded domain D with sufficiently smooth boundary S. By methods of J. Schauder [Math. Z. **38** (1934), 257–282; Studia Math. **5** (1934), 34–42] estimates are obtained in $D+S$ for the first derivatives of solutions, and for the Hölder constants of these first derivatives, in terms of bounds for the absolute magnitudes of the solution in $D+S$, the coefficients in the equations and their Hölder constants, and the values on S of the first derivatives of the solution and of the Hölder constants for these first derivatives. Estimates in $D+S$ more closely analogous to Schauder's, which contain only prescribed boundary data, are promised for a subsequent paper. Estimates of Schauder's type on closed subdomains of D are also given; these are a special case of the general results of Douglis and Nirenberg [Comm. Pure Appl. Math. **8** (1955), 503–538; MR **17**, 743]. Finally, by methods of C. B. Morrey [Trans. Amer. Math. Soc. **43** (1938), 126–166] and L. Nirenberg [Comm. Pure Appl. Math. **6** (1953), 103–156, 395; MR **16**, 367], the Hölder constants of a solution are estimated in any closed subdomain of D in terms of bounds for the absolute values of the coefficients and for the integral of the square of the solution and, furthermore, are estimated in $D+S$ by these quantities and also by bounds for the absolute values of the solution and of its first derivatives on S.

A. Douglis (New York, N.Y.).

Magenes, Enrico. Osservazioni su alcuni teoremi di completezza connessi con i problemi misti per le equazioni lineari ellittiche. Boll. Un. Mat. Ital. (3) **10** (1955), 452–459.

Sia $E(w)$ un operatore ellittico del secondo ordine in m variabili con coefficienti sufficientemente regolari in un dominio D. Se $\{w\}$ è l'insieme delle soluzioni dell'equazione $E(w)=0$, che sono di classe 1 in D e che hanno derivata conormale du/dn nulla su di una porzione aperta S_2 della frontiera S di D, i valori assunti dalle w su $S_1=S-S_2$ costituiscono un sistema di funzioni completo nello spazio delle funzioni di quadrato sommabile su S_1 sempre che sia identicamente nulla in D ogni funzione di $\{w\}$ nulla su S_1. Se $m=2$ lo stesso sistema è completo anche nello spazio delle funzioni continue su S_1. Questo interessante

risultato può essere utilizzato per la dimostrazione dei teoremi esistenziali relativi al problema: $E(u)=0$ in D, $du/dn=0$ su S_2, $u=0$ su S_1. L'A. segnala che per altra via G. Fichera ha dimostrato anche per $m>2$ la completezza dello stesso sistema nello spazio delle funzioni continue su S_1 [Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955; MR 17, 626]. *C. Miranda.*

Lopatin'skiĭ, Ya. B. Conditions for existence of the solution of the first boundary problem for a system of linear differential equations of second order and elliptic type. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 5–9. (Ukrainian. Russian summary)

Alternative de Fredholm dans le problème de Dirichlet pour des systèmes elliptiques de 2ème ordre dans des ouverts bornés. *J. L. Lions* (Nancy).

Louhivaara, Ilppo Simo. Über das erste Randwertproblem für die Differentialgleichung $u_{xx}+u_{yy}+qu+f=0$. Ann. Acad. Sci. Fenn. Ser. A. I. no. **183** (1955), 33 pp.

The treatment of the equation $u_{xx}+y_{yy}+qu+f=0$ for a bounded domain in the xy-plane, as contained in Courant and Hilbert, Methoden der mathematischen Physik [Bd. 2, Springer, Berlin, 1937], is revised to include the case where the real-valued function q is not necessarily $\leqq 0$. The revision is based on the method of orthogonal projections. [For more general results see, e.g., F. Browder, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 230–235, 741–747; MR **14**, 174, 473.] *A. N. Milgram.*

Louhivaara, Ilppo Simo. Über das zweite und dritte Randwertproblem für die Differentialgleichung $u_{xx}+u_{yy}+qu+f=0$. Ann. Acad. Sci. Fenn. Ser. A. I. no. **203** (1955), 14 pp.

This paper is a sequel to the one reviewed above.

A. N. Milgram (Minneapolis, Minn.).

★ **Bers, L.; and Nirenberg, L. On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 111–140. Edizioni Cremonese, Roma, 1955. 3000 Lire.

This paper deals with first-order elliptic systems for two functions of two independent variables, written in complex notation as

$$w_{\bar{z}}=\mu w_z+\nu\bar{w}_{\bar{z}}+\alpha w+\beta\bar{w}+j.$$

The uniform ellipticity of this system is expressed by $|\mu|+|\nu|\leqq k<1$. Otherwise the coefficients are assumed to be bounded and measurable. Dropping continuity is useful in various specific problems, linear and nonlinear. The main result of this paper is this representation theorem: Any solution of such an equation is of the form

$$w(z)=e^{s(z)}f[\chi(z)]+s_0(z),$$

where s and s_0 are Hölder-continuous functions real on the boundary of the underlying simply connected domain, χ is a Hölder continuous homeomorphism, and f analytic; s, s_0 and χ depend on the solution w, but not their Hölder modulus of continuity. The proof depends on approximation by equations with smoother coefficients, the Lichtenstein-Korn result, the uniformization theorem, an a priori estimate for the solution of a special boundary-value problem and the Ahlfors-Morrey-Lavrentieff lemma. The representation theorem shows that solutions of such equations behave locally like analytic functions. The a priori estimates contained in the representation theorem permit a very direct treatment of the Dirichlet problem for linear and nonlinear equations, and an extension of the Riemann mapping theorem along the lines of the work of Z. Schapiro [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **30** (1941), 690–692; MR **2**, 359]. *P. D. Lax.*

★ **Bers, L.; and Nirenberg, L. On linear and non-linear elliptic boundary value problems in the plane.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 141–167. Edizioni Cremonese, Roma, 1955. 3000 Lire.

In this paper the Dirichlet and Neumann problems are discussed for a single second-order equation in two independent variables:

$$A_{11}\phi_{xx}+2A_{12}\phi_{xy}+A_{22}\phi_{yy}+A_1\phi_x+A_2\phi_y+A_0\phi=B,$$

$A_{11}A_{22}-A_{12}^2\equiv 1$, $A_0\leqq 0$; the coefficients are bounded and measurable functions of their arguments in the linear case, continuous in the nonlinear case. The second derivatives of the solution are taken in the strong sense. The proof of existence is based on an a priori estimate of the absolute value of the solution, its first derivatives, the Hölder modulus of its first derivative and the square integral of the second derivatives over compact subdomains in terms of the bound on the absolute value of the coefficients, the Hölder modulus of continuity of the tangential or normal derivative respectively of a solution, and the value of the solution at a given point. The estimate for the absolute value of the solution is established by the strong-maximum principle for the Dirichlet problem, and by a Harnack-type theorem for the Neumann problem; the classical methods for proving these principles no longer apply in this case where the coefficients are discontinuous. The rest of the estimates follow from the Fatou-Korn-Privaloff theorem (linking the Hölder continuity of the real part of an analytic function to that of its imaginary part) with the aid of the representation theorem for the gradient of a solution established in the previous paper.

The boundary-value problem is then solved in the linear case by approximating the coefficients by smooth functions, in the quasilinear case by application of the Birkhoff-Kellogg-Schauder fixed-point theorem (in the C_1 norm). The necessary compactness in both cases follows from the a priori estimate stated before. Finally the boundary-value problems for nonlinear equations are treated by the continuity method.

The exposition in this paper and the one reviewed above is extremely lucid. *P. D. Lax* (New York, N.Y.).

Nitsche, Johannes. Über die linearen Randwertprobleme eines quasilinearen elliptischen Differentialgleichungssystems. Math. Z. **61** (1954), 336–347.

The author considers a system of Beltrami type

$$(1)\qquad Wu_x=-Fv_x+Gv_y,\ Wu_y=-Gv_x+Fv_y,$$
$$W=(EG-F^2)^{\frac{1}{2}}>0,$$

which is quasilinear, $E=E(x, y, u, v)$, $\cdots$, $G=G(x, y, u, v)$, and seeks a solution $w=u+iv$ in $|z|=|x+iy|\leqq 1$, satisfying the boundary conditions

$$(2)\qquad R^{\sigma}(u, v)\equiv \mathrm{Re}(e^{-i\sigma(\theta)}w)=h(\theta) \text{ on } |z|=|e^{i\theta}|=1.$$

Here σ and h are given real functions in $0\leqq\theta\leqq 2\pi$, with Hölder-continuous first derivatives; h has period 2π, while $\sigma(\theta+2\pi)=\sigma(\theta)+2\pi n$ for some integer n. In case $n\geqq 0$ one

requires in addition (2a): that on $|z|=1$ the Fourier components $\int_{|z|=1} e^{ij\theta}\,\mathrm{Im}(e^{-i\omega(\theta)}w)d\theta$ take on given values x_j for $0\leqq j\leqq n$. In case $n<0$, instead of (2) one requires, on $|z|=1$, merely (2b): that $R^{\sigma}(u, v)$ equal $h(\theta)$ modulo a linear combination of the functions $e^{ij\theta}$, $0\leqq j\leqq -(n+1)$.

Under certain differentiability requirements on the E, F, G, and under the assumption (3): $(E+G)/W$ is uniformly bounded for all $|z|\leqq 1$ and all values of u and v, the author, using Schauder's fixed-point theorem, shows the existence and uniqueness of a solution w having Hölder-continuous first derivatives, and satisfying the boundary conditions (2), (2a) in case $n\geqq 0$, or, in case $n<0$, or, in case $n<0$, the condition (2b). In case $n=0$ the author claims not to require assumption (3), but his argument seems incorrect (for $\lambda'=0$ his inequality (6), and hence (37), does not hold). L. Bers and the reviewer have also treated boundary-value problems for general quasi-linear elliptic systems [see the paper reviewed above].

L. Nirenberg (New York, N.Y.).

★ **Borkmann, Karl; und Oberländer, Siegfried. Lösung des allgemeinen Randwertproblems für eindimensionale gedämpfte Wellen bei harmonischem Zeitgesetz.** Akademie-Verlag, Berlin, 1955. 99 pp. DM 12.00.

Let $u(x, t)=C(x)\cos\omega t+S(x)\sin\omega t$ represent the periodic solution of the partial differential equation $u_{xx}-Au_{tt}=Du_t$ satisfying the boundary conditions $a_i u(x_i, t)+b_i u_x(x_i, t)=c_i\cos\omega t+s_i\sin\omega t$ $(i=1, 2)$, where the ten coefficients A, D, a_i, b_i, c_i and s_i are prescribed constants, A and D being non-negative. A systematic procedure for determining the functions $C(x)$ and $S(x)$ corresponding to a given set of numerical values of the ten coefficients is presented here. The authors begin with the fact that $C(x)$ and $S(x)$ are linear combinations of products of hyperbolic and circular sines and cosines like $\cosh\mu x\cos\nu x$. Numerical values of the four functions $F_1=\cosh k\alpha\cos\alpha$, $F_2=\sinh k\alpha\sin\alpha$, $F_3=\sinh k\alpha\cos\alpha$, $F_4=\cosh k\alpha\sin\alpha$, of some of their derivatives and certain of their combinations, are tabulated for values of the two variables k and α. The tables occupy nearly fifty pages. Properties of the four functions F_i are also listed. Simple formulas in the text together with numerical values from the tables give the coefficients in the linear combinations that represent the functions $C(x)$ and $S(x)$. *R. V. Churchill.*

Franklin, Joel; and Keller, Herbert B. A priori bounds for temperature in circulating fuel reactors. Quart. Appl. Math. **14** (1956), 57–62.

In the non-linear boundary value problem $u_t=u_{xx}+A(v)u$, $v_t+cv_x=u$ $(0\leqq x\leqq a, t\geqq 0)$, $u(0, t)=u(a, t)=0$, $v(0, t)=0$, $u(x, 0)=u_0(x)$, $v(x, 0)=v_0(x)$, c is a positive constant and the prescribed functions A, u_0 and v_0 as well as the functions u and v are assumed to satisfy conditions of regularity that will not be stated here. The functions u and v represent neutron density and temperature, respectively, in a certain type of reactor. It is proved that the function $v(x, t)$ is bounded $(t\geqq 0)$. A preliminary to that main result is that $u(x, t)\geqq 0$. As a consequence of the main result, a boundedness property of a norm of $u(x, t)$ is found. The norm here is the line integral $\int_{L_t} u\,ds$, where L_t is the segment of the straight line with slope $1/c$ through the point (a, t) included within the strip $0\leqq x\leqq a$, $t\geqq 0$. *R. V. Churchill.*

Moisil, Gr. C. Sur l'équation des ondes. Com. Acad. R. P. Romîne **5** (1955), 1421–1423. (Romanian. Russian and French summaries)

L'Auteur considère le problème de Cauchy pour l'équation des ondes $\partial^2\psi/\partial x^2-c^{-2}\partial^2\psi/\partial y^2=0$, lorsque les conditions initiales sont données: 1) sur un cercle parallèle d'un cylindre; 2) sur une génératrice d'un cylindre; 3) sur un cercle parallèle d'un tors. *Resumé français.*

Zeragiya, P. K. On solution of boundary problems for equations of parabolic type by the method of potentials. Soobšč. Akad. Nauk Gruzin. SSR **15** (1954), 569–573. (Russian)

Let

$$L(u)\equiv\sum\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)-\frac{\partial u}{\partial t}=0$$

be a parabolic equation defined over a domain T with Liapounoff boundary S in Euclidean 3-space. The first and second partial derivatives of the coefficients $a_{ij}(x)$ are assumed to be Hölder continuous in T. The author expresses the solution of the Dirichlet initial-value problem: $u(x, 0)=0$ $(x\in T)$; $u(\xi, t)=\phi(\xi, t)$ $(\xi\in S, t>0)$ in the form $u(x, t)=\int\mu(\xi, \tau)\partial_\xi\Gamma(x, t, \xi, \tau)$, where the integration is over $S\times(0t)$, where Γ is the fundamental solution in T [F. Dressel, Duke Math. J. **7** (1950), 186–203; **13** (1946), 61–70; MR **2**, 204; **7**, 450] and $\delta_\xi=\sum\cos(n, \xi_i)\sum a_{ij}\partial/\partial\xi_j$. Using the jump relations at the boundary the author derives a Volterra equation for $\mu(\xi, t)$. The same general procedure is valid for the Neumann problem, where $u(x, 0)=0$, and $\delta^1 u$ is given at the boundary. The results are generalizations of Tychonoff's [Bull. Univ. d'Etat Moscou. Ser. Internat. Sect. A. **1** (1938), nos. 8, 9] for the ordinary heat equation. *A. N. Milgram.*

Kikvidze, Z. A. On a system of partial differential equations of mixed type. Soobšč. Akad. Nauk Gruzin. SSR **15** (1954), 321–325. (Russian)

The author considers the system

$$(*)\qquad \frac{\partial u}{\partial x}+\operatorname{sgn} y\,\frac{\partial v}{\partial y}=0,\quad \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$$

which is of mixed elliptic-hyperbolic type. Let D be a Tricomi domain bounded by σ the arc in the elliptic portion and the characteristics L_1 and L_2. The boundary-value problem is solved for the system (*) in which u is prescribed on σ and L_1. This continues the work initiated by Lavrent'ev and Bicadze [Dokl. Akad. Nauk SSSR (N.S.) **70** (1950), 373–376; MR **11**, 724].

M. H. Protter (Berkeley, Calif.).

Wuang, Kuang-Ying. Sur l'unicité du problème de Tricomi de l'équation de Chaplygin. Acta Math. Sinica **5** (1955), 455–461. (Chinese. French summary)

In proving the uniqueness theorem for the Chaplygin equation with Tricomi data, $L(u)\equiv K(y)u_{xx}-u_{yy}=0$, the author uses the integral expression:

$$I\equiv\iint_{E+H}xuL(u)\,dxdy-4\iint_H\frac{1}{K'}x((-K)^{1/2}u_x-Ku_y)L(u)\,dxdy,$$

where E and H are respectively the upper (elliptic) and lower (hyperbolic) part of the domain D; the left vertical supporting line of D is taken as the y-axis, so that $x>0$ holds in D. The author demonstrates by Green's formula that the integral expression I can be represented as a sum of non-positive integrals, provided that $2(-K)^{3/2}+xK'F(y)\geqq 0$ holds in H, where $F(y)=2(K/K')'+1$ is the

same function as that used by Frankl. Suppose $y=y_0<0$ is the first zero of odd order of $F(y)=0$, then the author's uniqueness theorem is valid in any domain D with the property that $y>y_0-\varepsilon$ holds in H. The $\varepsilon>0$ is determined by $F(y)$ and by the distance of H from the left supporting line (this result is apparently not symmetric with respect to the left and the right of the domain, due to the use of x in I). The author shows furthermore that under some condition on $F(y)$ there is a singularity of $K(y)$ between two zeros of $F(y)=0$. These results are closely related to an earlier paper by Ou and Ding on the same subject [same Acta **5** (1955), 393–399; MR **17**, 748].

Y. W. Chen (Berkeley, Calif.).

Marić, Vojislav. On the Green's function of the biharmonic operator. Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 59–66.

Let $G(P, Q; \lambda)$ be the Green's function of the equation $\Delta\Delta u-\lambda u=f$ with boundary conditions $u=\partial u/\partial\nu=0$ on a finite two-dimensional domain. One can write

$$(1)\qquad G(P, Q; \lambda)=\frac{1}{2\pi}R(P, Q; \lambda)-\gamma(P, Q; \lambda),$$

where R is a fundamental solution which has been explicitly written in terms of Bessel functions by A. Pleijel [Ark. Mat. Astr. Fys. **27A** (1940), no. 13; MR **2**, 291]. In the cited paper Pleijel, in the course of finding asymptotic distributions for the eigenvalues and eigenfunctions of the clamped plate, has given a bound for $|\gamma(P, P, -\lambda)|$ for positive real λ. His result was sharpened and extended to $|\gamma(P, Q, -\lambda)|$ by Bojanić and Vučković [Srpska Akad. Nauka. Z. Rad. **35**, Mat. Inst. **3** (1953), 107–128; MR **16**, 482].

By extending the tools and results of Pleijel the author finds the following sharper estimate, which is good for complex λ with $|\arg\lambda|<\pi$.

$$(2)\quad |\gamma(P, Q; -\lambda)|\leqq ml_p^{-15/2}|\lambda|^{-5/8}|\sin\tfrac{1}{2}(\theta-\pi)|^{-1}\Omega(\lambda, l_p),$$

where m is a constant depending only on the domain, l_p is the distance of P from the boundary, $\theta=\arg\lambda$, and

$$(3)\quad (\Omega\lambda, l_p)=\exp[-\tfrac{1}{2}l_p\,\mathrm{Re}(\lambda^{\frac{1}{2}}e^{-\pi i})]+\exp[-\tfrac{1}{2}l_p\mathrm{Re}\,\lambda^{\frac{1}{2}}e^{\pi i}].$$

There appear to be some misprints in the paper with a particularly high concentration on the last page. This casts some doubt on the accuracy of (2). In fact, assuming that some of the intermediate results are correct, the reviewer finds $-7/2$ rather than $-15/2$ for the exponent of l_p. Furthermore, it appears that the proof of (2) is only valid for $|\lambda|$ bounded away from zero.

H. F. Weinberger (College Park, Md.).

Massimi, Rosanna. Un teorema di esistenza per i problemi di Cauchy con condizioni iniziali su curve dotate di rappresentazione cartesiana. Rend. Mat. e Appl. (5) **13** (1955), 433–439.

The author shows that there is at least one function $u=u(x, t)$ satisfying the following conditions

$$\frac{\partial^m u}{\partial t^m}+a\frac{\partial^n u}{\partial x^n}=0,\ x_1<x<x_2,\ |t-\alpha(x)|<l,$$

$$\left[\frac{\partial^i u}{\partial t^i}\right]_{t=\alpha(x)}=\phi_i(x)\ (i=0, 1, \cdots, m-1),$$

if $\phi_i(x)$, $\alpha(x)$ meet the conditions listed below.

$$\left|\frac{d^h\phi_i(x)}{dx^h}\right|<N(h!)^{m/n}/\varrho^h,\ x_1\leqq x\leqq x_2,\ h=0, 1, 2, \cdots,$$

$$\left|\frac{d^h\alpha(x)}{dx^h}\right|<l(h!)^{m/n}/\varrho^h,\ x_1\leqq x\leqq x_2,\ h=0, 1, 2, \cdots.$$

Here $a, l>0$, $N>0$, $\varrho>0$ are constants and it is assumed that $\varrho^n>|a|l^m2^{2m+n}$. As a function of t the solution $u(x, t)$ has m continuous derivatives and as a function of x belongs to class C^∞. *F. G. Dressel* (Durham, N.C.).

See also: Žautykov, p. 931; Lagerstrom and Cole, p. 1021.

Integral Equations, Equations in Infinitely Many Variables

Börsch-Supan, Wolfgang. Eigenfunktionen einer zusammengesetzten Eigenwertaufgabe als Reihen nach den Eigenfunktionen sämtlicher Teilaufgaben. Der Stahlbau **24** (1955), no. 3, 62–63.

This note discusses the relationship between the eigenvalues and eigenvectors of the homogeneous linear integral equation of the second kind

$$w(s)=\lambda\int K(s, t)w(t)dt$$

with the eigenvalues and eigenvectors of the related integral equations the sum of whose kernels $M(s, t)+N(s, t)=K(s, t)$, where M, N, K are real, non-zero, continuous, and symmetric. *H. Polachek.*

Sparenberg, J. A. Application of the theory of sectionally holomorphic functions to Wiener-Hopf type integral equations. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 29–34.

The author considers homogeneous and non-homogeneous integral equations whose form is of the Wiener-Hopf type but which do not obey the hypothesis that there is a common strip of analyticity for the required Fourier transforms. It is shown that it is possible to solve such equations if the necessary transforms exist on a line parallel to the axis of the reals. Two examples are given.

A. E. Heins (Pittsburgh, Pa.).

Citlanadze, È. S. Investigation of a class of nonlinear integral equations by a topological analogue of direct methods. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **21** (1955), 125–143. (Russian)

Consider the integral equation:

$$f(r, x, I_1, \cdots, I_n, \cdots)=f\Big[r, x(r), \int_0^1 K_1(r, t_1, x(t_1))dt_1, \cdots,$$

$$\int_0^1\cdots\int_0^1 K_n(r, t_1, \cdots, t_n, x(t_1), \cdots, x(t_n))dt_1\cdots dt_n, \cdots\Big]$$

$$=\lambda|x(r)|^{p/q}\operatorname{sign}x(r),$$

where $x(r)\in S_1$, the unit sphere of $L_p(0, 1)$; the $K_1, \cdots, K_n, \cdots$, and f are given functions; and the operator $f(x)=f(r, x, I_1, \cdots, I_n, \cdots)$ takes $x(r)$ into L_q $(1/p+1/q=1)$. If $\|x(r)\|=1$, then $x(r)$ is called an eigenfunction with eigenvalue λ of operator $f(x)$. Assume f satisfies the following conditions: 1) f has a strong differential $df_x(h)$ at each point x of S_1; for h_1, $h_2\in L_p$ $(df_x(h_1), h_2)=(df_x(h_2), h_1)$; and $(df_x(h_1), h_2)$ is strongly continuous at each point $x\in S_1$; 2) there exists a linear, bounded, weakly continuous (i.e., weakly convergent sequences are taken into strongly convergent sequences) operator $U(x)$ defined in S_1 and a positive constant C such that for x_1, $x_2\in S_1$, $\|f(x_1)-f(x_2)\|\leqq C\|U(x_2-x_1)\|$; 3) $f(\theta)=0$ and $f(x)>0$ for $x\neq\theta$ where θ is the zero of L_p; 4) $(f(x), x)>0$ for $x\neq\theta$. Using an extended from of the Lyusternik-Šnirelman theory of category developed in a previous paper [Trudy

Moskov. Mat. Obšč. 2 (1953), 235–274; MR 14, 1094], the author proves that $f(x)$ has a countably infinite set of eigenfunctions x_ν with corresponding eigenvalues $\lambda_\nu \neq 0$. The method of proof is much like that in another paper of the author [Akad. Nauk Gruzin. SSR. Trudy Tbliss. Mat. Inst. Razmadeze 20 (1954), 245–278; MR 16, 934].

J. Cronin (Southbridge, Mass.).

Vaĭnberg, M. M. The operator of V. V. Nemyckiĭ. Ukrain. Mat. Ž. 7 (1955), 363–378. (Russian)

If $g(u, x)$ is a real-valued function defined for u in euclidean m-space and x in a region B of n-space, then g defines a function h, here called the Nemyckiĭ operator, carrying vector-valued functions $u(\cdot)$ defined on B to other such functions, by the equation $[hu](x)=g(u(x), x)$. The author shows that a condition already shown by him to be sufficient (with some continuity on g) that h carry some L^p continuously into an L^{p_1} [Dokl. Akad. Nauk SSSR (N.S.) 73 (1950), 253–255; MR 12, 111] is necessary that h carry L^p into L^{p_1}. An example shows that h can be continuous without being uniformly continuous. Conditions are given under which the Gateaux differential of h can be calculated in terms of the operator h' defined from the partial derivative g_u. The last section returns to the subject of another earlier paper [ibid. 80 (1951), 309–312; MR 13, 353] on the Hammerstein operator and its gradient.

M. M. Day (Urbana, Ill.).

Cooke, J. C. A solution of Tranter's dual integral equations problem. Quart. J. Mech. Appl. Math. 9 (1956), 103–110.

The author's summary as follows is "Tranter [same J. 3 (1950), 411–419; 7 (1954), 317–325; MR 12, 713; 16, 372] has given the solution of the pair of integral equations (1) and (2) below in the form of a Neumann series of Bessel functions whose coefficients are determined by means of an infinite series of linear equations. This paper gives the formal solution as an integral, containing an unknown function which is determined by means of an integral equation of Fredholm's type. This solution appears to have both theoretical and practical advantages over Tranter's solution in that it is (in a sense) in closed form; moreover it does not depend on certain parameters being specified as large or small in order to complete a practical solution. Applications are given to several potential problems involving circular disks, some of these problems being believed to be new."

A. E. Heins.

Èĭdel'man, S. D. On an integral equation with nonregular kernel. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 1(67), 235–239. (Russian)

The integral equation $\mu(t)+2A^{(1)}\mu(t)=\phi(t)$, in which

$$2aA^{(m)}u(t)=\int_0^t km^{-1/2}(t-\tau)^{-3/2}\exp\left[ik^2/4m(t-\tau)\right]u(\tau)d\tau,$$

$a=(2\pi)^{\frac{1}{2}}(1+i)$, $k>0$, $m>0$, is found by iteration to have the solution

$$\mu(t)=2\sum_{n=1}^{\infty}(-1)^n A^{(n\nu)}\phi(t)+\phi(t),$$

provided ϕ has a continuous second derivative and satisfies the condition $\phi(0)=0$. The result is based upon the composition formula $2A^{(m)}A^{(n)}=A^{(p)}$, $p=mn/(m^{\frac{1}{2}}+n^{\frac{1}{2}})^2$, established in the paper.

R. N. Goss.

Krukovskiĭ (Krukovskiĭ-Sinevič), B. V. On the relation between some formulas of the determinantal and nondeterminantal theories of systems of linear equations with an infinite number of unknowns. Kiev. Avtomobil.-Dorož. Inst. Trudy 2 (1955), 176–188. (Russian)

The infinite system of equations

$$(*)\qquad x_i+\sum_{k=1}^{\infty}A_{ik}x_k=c_i\quad(i=1,2,\cdots)$$

for the conditions

$$(**)\qquad \sum_{i,k=1}^{\infty}|A_{ik}|<\infty,\ \sum_{i=1}^{\infty}|c_i|<\infty,$$

coefficient determinant $D\neq 0$, is known to have a solution given in the Cramer form $x_i=D^{(i)}/D$ (extented to the case of infinite determinants). On the other hand, the conditions (**) are sufficient for the non-determinantal method of Schmidt [Rend. Circ. Mat. Palermo 25 (1908), 53–77] to apply. The author shows conversely how from the determinantal method, by a series of elementary processes, one can obtain the Schmidt solution (still under conditions (**)). Some lightening of conditions (**) also is possible.

I. M. Sheffer.

See also: Gabibzade, p. 956; Haplanov, p. 972; Lévy, p. 978.

Theory of Probability

Vorovič, I. I. On stability of motion for random disturbances. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 17–32. (Russian)

The author considers systems of differential equations

$$(*)\dot{x}_i=\sum_{j=1}^{n}a_{ij}x_j+\mu\sum_{k=n+1}^{m}b_{ik}x_k+\mu\varphi_i(x_1,\cdots,x_n;\ t;\ x_{n+1},\cdots,x_m)\quad(i=1,\cdots,n),$$

where $(x_{n+1},\cdots,x_m)$ is a stochastic process all of whose moments are given functions of the time parameter t, the coefficients a_{ij} and b_{ik} are constant, μ is a real parameter, and φ_i are analytic in a closed neighbourhood $\bar{G}$ of the origin and continuous in t for $0\leq t\leq T$. It is assumed that the distribution of $(x_{n+1},\cdots,x_m)$ is such that the solutions of (*) (presumably satisfying suitable initial conditions) remain in G for $0\leq t\leq T$ and can always be expanded into uniformly convergent power series in μ for $0\leq\mu\leq 1$. Formal expansions are obtained giving all the univariate and mixed moments of $x_1,\cdots,x_m$ as power series in μ, whose coefficients depend on the moments of $x_{n+1},\cdots,x_m$. An alternative approach based on the method of successive approximations is also outlined. In the case of time-dependent coefficients, the existence of finite moments of $(x_1,\cdots,x_n)$ is discussed in connection with the stability of the corresponding determinate system. Finally the author states that if the right-hand sides of (*) do not contain t explicitly and if $(x_{n+1},\cdots,x_m)$ is stationary in his sense, i.e. has all its moments independent under time shifts, then (*) has a solution which is stationary in the same sense; the proof consists of calculating the moments of this solution.

S. K. Zaremba (Wolverhampton).

Sato, Hiroshi. Information theory and statistical mechanics. Rep. Univ. Electro-Commun. 7 (1955), 41–47. (Japanese. English summary)

Romanovskiĭ, V. I. Search for and estimate of nonsymmetry of linear and plane eccentricity. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 16–21. (Russian)

Let e be the distance between the centers of two intervals (circles). The author considers the problem of estimating e if the smaller interval (circle) is superimposed on the larger interval (circle) by a certain chance mechanism. *G. E. Noether* (Boston, Mass.).

Lévy, Paul. Intégration d'une équation intégrale non linéaire. C. R. Acad. Sci. Paris **242** (1956), 1252–1255.

The author outlines a method of solving the equation

$$\Gamma(t_1, t_2)=\int_0^t F(t_1, u)F(t_2, u)du,$$

where $t=\min(t_1, t_2)$, for the unknown function F. The equation is suggested by the fact that Γ is the covariance function of the stochastic process defined by $\Phi(t)=\int_0^t F(t, u)dy(u)$, where the $y(u)$ process is the Brownian-motion process. The symmetry and positive-definiteness properties appropriate to this setting are imposed on Γ. As an example, the case in which Γ and F are homogeneous polynomials is discussed completely. The possibility of generalizing the discussion by the use of Schwartzian distribution notions, replacing the function F by a more general concept, is indicated. *J. L. Doob.*

Lévy, Paul. Fonctions aléatoires à corrélation linéaire. C. R. Acad. Sci. Paris **242** (1956), 1575–1578.

Lévy, Paul. Fonctions aléatoires à corrélation linéaire. C. R. Acad. Sci. Paris **242** (1956), 2095–2097.

A stochastic process $\{\Phi(t), t>0\}$ is said to have linear correlation if, for each pair t, t' with $t<t'$, $\Phi(t')$ can be expressed in the form $U+V$, where U is a linear functional of the values of $\Phi(u)$ for $u \in (0, t]$, and V is independent of these values. If this condition is made more restrictive by replacing $(0, t]$ by any set not containing t', then the process is said to have linear complete correlation. Conditions necessary and sufficient for linear complete correlation are indicated. The author studies processes of the form

$$(*) \qquad \Phi(t)=\int_0^t F(t, u)dX(u)+\int_0^t G(t, u)dY(u),$$

where the X and Y processes are mutually independent and additive (independent increments), the X process is Laplacian with continuous sample functions, and the Y process is a jump process. In important cases such processes have linear correlation. There may be many possible representations of $\Phi(t)$ in this form. A representation is called canonical if U as defined above is given by the right side of (*) with t in F and G replaced by t'. Conditions under which there is a canonical representation are discussed, as well as conditions under which F and G can be determined knowing the covariance function of the $\Phi(t)$ process and the variance functions of the X and Y processes. The solution of an integral equation involved in the latter problem was discussed in the paper reviewed above. Some errors in the first paper are corrected in the second. *J. L. Doob* (Urbana, Ill.).

Gulotta, Beniamino. Su alcune questioni riguardanti gli eventi compatibili e sulla loro applicazione a una questione di logica. Giorn. Ist. Ital. Attuari **17** (1954), 54–61.

The author shows that if $H_1, H_2, \cdots$ is a set of events and $T_1, T_2, \cdots$ is a corresponding set such that the T_i's are disjoint and each T_i implies at least one of the H_j's and each H_i implies the corresponding T_i then each T_i implies the corresponding H_i. He considers both strict implications and implications interpreted as meaning that the corresponding conditionals have probability 1. This is related to the fact if T implies H then the negation of H implies the negation of T. *A. H. Copeland, Sr.*

Huron, Roger. Sur une transformation linéaire des fréquences observées dans les grands échantillons non exhaustifs, de taille fixée, et extraits au hasard d'une urne à k catégories. C. R. Acad. Sci. Paris **242** (1956), 1951–1953.

Wooding, R. A. The multivariate distribution of complex normal variables. Biometrika **43** (1956), 212–215.

The author shows that if the normally distributed real and imaginary components of the complex variate $v_n=x_n+iy_n$ satisfy the covariance relations $E(x_mx_n)=E(y_my_n)$, $E(x_my_n)=-E(x_ny_m)$, then the distribution of N complex variates v_n has the form

$$\pi^{-N}|L|^{-1}\exp(-V'^*L^{-1}V),$$

where V is the complex column vector $V=X+iY$ formed from the $v_n=x_n+iy_n$, and L is the Hermitian covariance matrix $L=E(VV'^*)=2(A-iB)$, $A=E(XX')=E(YY')$, $B=E(XY')=-B'=-E(YX')$. *S. Kullback.*

Prohorov, Yu. V. On sums of identically distributed random quantities. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 645–647. (Russian)

Let F^n be the distribution function of the sum of n mutually independent random variables with a common distribution function which does not change with n. It is proved that there is a sequence of infinitely divisible distribution functions, $\{G_n\}$, for which $\sup_x|F^n(x)-G_n(x)|\to 0$. It is stated that, if the common distribution function has an absolutely continuous component or has no continuous component, the conclusion can be strengthened to $\mathrm{Var}(F^n-G_n)\to 0$. *J. L. Doob* (Urbana, Ill.).

Gnedenko, B. V. Limit theorems for sums of independent summands and for Markov chains. Ukrain. Mat. Ž. **6** (1954), 5–20. (Russian)

Historical summary. *J. L. Doob* (Urbana, Ill.).

Manevič, D. V. On a local limit theorem for stationary Markov chains. Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 7, 5–9. (Russian. Uzbek summary)

Let s_k be the sum of the first $k+1$ random variables of a Markov chain with finitely many states and stationary transition probabilities. It is supposed that the summands are integer-valued. Let s_k have expected value A_k and variance B_k^2, and define $z_{km}=(m-A_k)/B_k$. Then it is proved that, as $k\to\infty$,

$$[B_kP\{s_k=m\}-(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}z^2_{km})]\to 0$$

uniformly as m varies if and only if the highest common factor of the differences between elements in the range of values of the summands is 1. The result is proved by means of an application of the corresponding result for sums, of mutually independent random variables due to Gnedenko [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 3(25), 187–194; MR **10**, 132]. *J. L. Doob* (Urbana, Ill.).

Petrov, V. V. Extension of Cramér's limit theorem to nonidentically distributed independent quantities. Vestnik Leningrad. Univ. **1953**, no. 8, 13–25. (Russian)

Let $\{Z_n, n\geq 1\}$ be a sequence of mutually independent random variables, with $E\{Z_n\}=0$, $\sum_1^n E\{Z_j^2\}=B_n$. Let F_n be the distribution function of $\sum_1^n Z_j/B_n^{\frac{1}{2}}$, and let Φ be the normal distribution function (mean 0, variance 1). The following conditions are imposed. 1. For some positive numbers k, K, A, $k\leq|E\{\exp(hZ_j)\}|\leq K$, if $j\geq 1$ and $|h|<A$. 2. $\inf_n B_n/n>0$. 3. For every $\varepsilon>0$ there is a constant $C>1$ for which, for all n,

$$\frac{1}{n}\sum_1^n\int_{-\infty}^{-c} y^2 dP\{Z_j\leq y\}<\varepsilon.$$

Now suppose that x varies with n in such a way that $1<x=o(n^{\frac{1}{2}}/\ln n)$. Then it is proved that

$$\frac{1-F_n(x)}{1-\Phi(x)}=\left[\exp\left\{\frac{x^3}{n^{\frac{1}{2}}}\lambda_n\left(\frac{x}{n^{\frac{1}{2}}}\right)\right\}\right]\left[1+O\left(\frac{x\ln n}{n^{\frac{1}{2}}}\right)\right]$$

and $F_n(-x)/\Phi(-x)$ is given by the same expression with x replaced by $-x$. Here $\lambda_n(\cdot)$ is, for each n, a power series in its argument, which converges, uniformly as n varies, in some neighborhood of the origin. This results was proved by Cramér [Les sommes et les fonctions de variables aléatoires, Hermann, Paris, 1938, pp. 5–23] for the case that the random variables have a common distribution function. Related results have also been proved by Feller [Trans. Amer. Math. Soc. **54** (1943), 361–372; MR **5**, 125]. *J. L. Doob* (Urbana, Ill.).

Bobrov, A. A. On the relative stability of sums of positive random quantities. Moskov. Gos. Univ. Uč. Zap. **145**, Mat. 3 (1949), 92–107. (Russian)

The sequence $\{\xi_n, n\geq 1\}$ of mutually independent nonnegative random variables is said to be relatively stable if there is a sequence $\{A_n, n\geq 1\}$ of positive constants such that $p\lim_{n\to\infty} S_n/A_n=1$, where $S_n=\sum_1^n \xi_j$. It is proved that, if ξ_j has distribution function F_j, the sequence is relatively stable if and only if there is a sequence $\{A_n, n\geq 1\}$ with the property that:

$$\sum_{k=1}^n\int_{|x-a_k|\geq\varepsilon A_n} dF_k(x)\to 0,\quad \frac{1}{A_n}\sum_{k=1}^n\int_{|x-a_k|<\varepsilon A_n} x\,dF_k(x)\to 1\ (n\to\infty),$$

where $a_k=A_k-A_{k-1}$. If the condition is satisfied, $\{A_n, n\geq 1\}$ can be taken as the normalizing sequence of the stability definition. Moreover, the sequence of random variables is relatively stable with the above prescribed normalizing sequence if and only if the stated limit relations are satisfied. If the sequence of random variables is relatively stable with the above normalizing sequence, then (*) $\lim_{n\to\infty} P\{\xi_k/A_n\geq\varepsilon\}=0$ uniformly for $k\leq n$ if and only if $A_n\to\infty$ and $A_{n-1}/A_n\to 1$. The previous work in this area has assumed that (*) is satisfied (the quotients in (*) are asymptotically constant in any case), and in this case the author had already found necessary and sufficient conditions for relative stability [Moskov. Gos. Univ. Uč. Zap. Mat. **15** (1939), 191–202]. It is shown that the sequence of random variables is strongly relatively stable (probability 1 limit instead of stochastic limit) under the hypothesis that each ξ_k has finite variance b_k, and with $A_n=\sum_{k=1}^n E\{A_k\}$, if $A_n\to\infty$ and if

$$\sum_{n=1}^\infty b_n/A_n^2<\infty.$$

J. L. Doob (Urbana, Ill.).

Tsurumi, Shigeru. On the strong law of large numbers. Tôhoku Math. J. (2) **7** (1955), 166–170.

The author shows that if $X_1, X_2, \cdots$ be independent random variables and if, for some constant K, each integer m, and each rectangle R in m-space,

$$(*)\quad \limsup_{n\uparrow+\infty} n^{-1}\sum_{0\leq i<n}\Pr(X_{i+1}, X_{i+2}, \cdots, X_{i+m})\in R)\leq K\Pr(X_1, X_2, \cdots, X_m)\in R),$$

then

$$(**)\quad \sum_{n>0} E(|X_n|, |X_n|\in A_n)<+\infty,$$

for some partition of $[0, +\infty)$ into Borel sets $A_1, A_2, \cdots$, is equivalent to the strong law of large numbers:

$$(***)\quad \Pr(n^{-1}\sum_{0<i\leq n} X_i\to \text{const})=1.$$

When $X_1, X_2, \cdots$ have the same distribution, (*) is automatic, the sum in (**) is $E(|X_1|)=E(|X_2|)=\cdots$, and the implication (**)$\Longleftrightarrow$(***) is the classical Kolmogorov criterion [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933]. *H. P. McKean, Jr.*

Hanš, O. The strong law of large numbers for generalized random variables. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 15–17.

The author considers random variables $S_1, S_2, \cdots$ with values in some Banach space B and states conditions, necessary and sufficient in order that S_n converge in the weak or strong (B) topology with (outer) probability one. Some applications are sketched, and conditions, equivalent to the strong law of large numbers, are stated, but these are incorrect. *H. P. McKean, Jr.*

Lipschutz, Miriam. On strong bounds for sums of independent random variables which tend to a stable distribution. Trans. Amer. Math. Soc. **81** (1956), 135–154.

The author obtains strong lower bounds for cumulative sums $S_n=X_1+X_2+\cdots+X_n$ of independent identically distributed positive continuous random variables X whose distribution function F lies within the domain of attraction of a stable law. Strong upper bounds have been given by P. Lévy [Studia Math. **3** (1931), 119–155], J. Marcinkiewicz [Bull. Sém. Math. Wilno **2** (1939), 22–34; MR **1**, 21] and W. Feller [Trans. Amer. Math. Soc. **67** (1949), 98–119; MR **11**, 255]. It is supposed that

$$1-F(x)\equiv\Pr\{X\geq x\}=h(x)/x^\gamma\quad (x>x_0),$$

where $0<\gamma<2$ and where $h(x)$ satisfies $\lim_{n\to\infty} h(cn)/h(n)=1$ for any positive constant c, together with certain other regularity conditions. The paper contains the following theorem valid when $0<\gamma<1$ (and a companion result for $1<\gamma<2$): Let $\psi(n)\uparrow\infty$ and let $\psi(n)/n\downarrow 0$; then

$$\Pr(S_n<b_n/[\psi(b_n)]^{1/\gamma}\text{ i.o.})=0\text{ or }1$$

according as

$$\int_1^\infty \frac{[\psi(x)]^{(1-\gamma)/2}}{x}\exp\{-k_\gamma[\psi(x)]^{1/(1-\gamma)}\}dx<\infty\text{ or }=\infty,$$

where $1-F(b_n)\equiv 1/n$ and

$$k_\gamma\equiv[\Gamma(1-\gamma)]^{1/(1-\gamma)}\{\gamma^{\gamma/(1-\gamma)}-\gamma^{1/(1-\gamma)}\}.$$

The method of proof rests sesentially on a lemma of K. L. Chung and P. Erdös [ibid. **72** (1952), 179–186; MR **13**, 567]. *D. G. Kendall* (Oxford).

Romanovskiĭ, V. I. On applications of infinite matrices to the theory of probability. Dokl. Akad. Nauk Uzbek. SSR. **1953**, no. 9, 3–6. (Russian. Uzbek summary)

The limit relation $x_n \to x$ implies that (1) $\sum_{k=1}^{\infty} a_{nk}x_k \to x$ under classical conditions on the matrix (a_{nk}). In particular, if the matrix has non-negative elements and row sums 1, there is a simple probability interpretation of (1). A second application is of a similar nature.

J. L. Doob (Urbana, Ill.).

Itô, Kiyosi. Spectral type of the shift transformation of differential processes with stationary increments. Trans. Amer. Math. Soc. **81** (1956), 253–263.

Let $X(t, \omega)$, $-\infty < t < \infty$, be a measurable differential process with stationary increments. Then the so called shift transformations $\{T_\tau\}$ form a one-parameter group of measure-preserving set transformations on the Borel system B_X of difference sets and induce a group $\{U_\tau\}$ of unitary operators on the L^2-space H_X over B_X. It is proved here that H_X is isomorphic to the direct sum of the complex numbers and countably many repetitions of the L^2-space over the real line in such a way that U_τ becomes the operator

$$\Phi_\tau\colon (c, f_1(t), f_2(t), \cdots) \to (c, f_1(t+\tau), f_2(t+\tau), \cdots).$$

Thus processes of this kind are all spectrally equivalent, a result that generalizes a theorem of Kakutani [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 319–323; MR **12**, 33] which pertained to the special case of Brownian motion.

G. Hufford (Stanford, Calif.).

Kozulyaev, P. A. On the theory of extrapolation of stationary sequences. Moskov. Gos. Univ. Uč. Zap. **145**, Mat. 3 (1949), 59–91. (Russian)

Let $\{y(t), -\infty < t < \infty\}$ (t integral) be an orthonormal sequence of random variables [the added condition imposed by the author that $Ey(t)=0$ seems to be irrelevant]. Define $x(t, m)=m^{-1}\sum_{i=1}^{m} y(t+i)$. The author evaluates explicitly, for each pair of natural numbers n, l, the best linear least-squares approximation of $x(t+l, m)$ by linear combinations of $x(t, m), \cdots, x(t+1-n, m)$, and also evaluates the corresponding mean-square error. The results in the case $l=1$ were stated without proof in a previous paper [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **30** (1941), 13–17; MR **3**, 4]. *J. L. Doob* (Urbana, Ill.).

Kiefer, J.; and Wolfowitz, J. On the characteristics of the general queueing process, with applications to random walk. Ann. Math. Statist. **27** (1956), 147–161.

It is shown that in the general queueing process certain sample moments converge, with probability one, to the population moments. Necessary and sufficient conditions for the finiteness of the latter, and for the moments of the distribution of the extreme distance in a random walk, are proved. *D. V. Lindley* (Cambridge, England).

Mihoc, G. La loi des évènements rares pour les chaînes de Markov. Science R. P. Roumaine **1** (1953), 14–18 (1954).

French version of an earlier Romanian paper [Acad. R. P. Române. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **4** (1952), 783–790; MR **15**, 635].

Rodeanu, R. Sur un théorème ergodique concernant les chaînes de Markov. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Ştiinţ. Nat. **4** (1955), no. 8, 39–42. (Romanian. Russian and French summaries)

Let $P(t_1, x, t_2, A)$ be the probability that a Markov process in state x at time t_1 will be in a state of the set A at time t_2. Suppose that there are times $t_1 < t_2 < \cdots \to \infty$ and corresponding values $\alpha(t_1), \alpha(t_2), \cdots$ with $\sum \alpha(t_j)=\infty$ such that, for any decomposition of the state space into two disjunct sets A_1, A_2, $\max_i \inf_x P(t_{j-1}, x, t_j, A_i)$, for all j. It is shown that then, for each t,

$$\lim_{s\to\infty} |P(\tau, x, s, A) - P(t, y, s, A)| = 0$$

uniformly as x, y, A vary. [This statement differs slightly from that of the author, which seems to be inexact.]

J. L. Doob (Urbana, Ill.).

Gihman, I. I. Markov processes in problems of mathematical statistics. Ukrain. Mat. Ž. **6** (1954), 28–36. (Russian)

Expository lecture, containing refinements of and new approaches to previously known results. *J. L. Doob.*

Ramakrishnan, Alladi; and Mathews, P. M. Stochastic processes associated with a symmetric oscillatory Poisson process. Proc. Indian Acad. Sci. Sect. A. **43** (1956), 84–98.

Let $m(t)$ be a generalized Poisson process, where events occur with rate λ, each event adding ± 1 to m with equal probabilities; $m(0)=1$. Distribution, first-passage times, and recurrence times for m are discussed. Let $y(t)=\int_0^t m(\tau)d\tau$; then y is not Markov, although the pair (y, m) is. Nevertheless, the frequency function $\pi(y, t)$ satisfies the "backward" equation

$$(1)\quad \partial\pi(y, t)/\partial t = -\lambda\pi(y, t) + \tfrac{1}{2}\lambda[\pi(y-t, t)+\pi(y+t, t)].$$

This is associated with the fact that $y(t)$ has, for fixed t, the same distribution as the Markov process $y^*(t)$, $t \geqq 0$, where y^* is a process of jumps, rate λ; a jump at τ has magnitude $\pm\tau$ with equal probabilities. If $F(y, t)dydt$ is the probability that y^* is in $(y, y+dy)$ for the first time in $(t, t+dt)$, then $F(y, t)=\frac{1}{2}\lambda\{\pi(y-t, t)+\pi(y+t, t)\}$, which can be evaluated using (1). A generalization of the above is applied to a model for multiple scattering. Three models for scattering in multiplicative processes are proposed.

T. E. Harris (Santa Monica, Calif.).

Onicescu, O. Chaînes continues à liaisons complètes. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Ştiinţ. Nat. **3** (1954), no. 4–5, 73–85. (Romanian. Russian and French summaries)

The author studies chains of complete connection [called chains of type *O-M* by Doeblin and Fortet, Bull. Soc. Math. France **65** (1937), 132–148] in the continuous-parameter case, under the assumption of a finite state space. A differential equation satisfied by the transition probabilities is found, generalizing the one relevant to the Markov case, and the ergodic properties of the transition probabilities are investigated. *J. L. Doob.*

Gel'fand, I. M.; and Yaglom, A. M. Methods of the theory of random processes in quantum physics. Vestnik Leningrad. Univ. **11** (1956), no. 1, 33–34. (Russian)

General remarks on integrals in function space and their physical interpretation. *J. L. Doob.*

Weiss, George H. A note on the coincidence of some random functions. Quart. Appl. Math. **14** (1956), 103–107.

This paper deals with a system capable of randomly attaining two states, "on", and "off". For each state there

is a known distribution for the length of time the system spends in the state. The question is, what is the expected amount of time that n identical such systems are in the same state simultaneously.

Calculating the probability of the system's being in a given state at time t leads to two simultaneous renewal equations, which are treated by Laplace transforms. [A more general problem is treated by Smith in Proc. Roy. Soc. London. Ser. A. **232** (1955), 6–31; MR **17**, 502.]

E. Reich (Santa Monica, Calif.).

Steyn, H. S. On the univariable series

$$F(t)\equiv F(a; b_1, b_2, \cdots, b_k; c; t, t^2, \cdots, t^k)$$

and its application in probability theory. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 190–197.

The series in the title is obtained from the multivariable series of hypergeometric type used by the author in an earlier paper [same Proc. **54** (1951), 23–30; MR **12**, 722] by the identification $t_i\equiv t^i$. Conditions for convergence are given, which qualify it for probability use when properly normalized. The normalized series with $t=1+\alpha$ is a factorial moment generating function and the first three factorial moments are obtained by direct expansion. The limiting form for large absolute values of the parameters is shown to be Gaussian. Special cases examined are: (1) the factorial multinomial function, which includes the hypergeometric distribution; (2) the negative factorial multinomial function, and (3) extensions of the Eggenburger-Polya distribution. *J. Riordan.*

See also: Kendall, p. 931; Goddard, p. 936; Aczél et Zubrzycki, p. 944; Kac, p. 944; Good, p. 981; Romanovskiĭ, p. 984; Dunford and Schwartz, p. 987; Reuter, p. 988; Ishida, p. 1030.

Mathematical Statistics

Anis, A. A. On the moments of the maximum of partial sums of a finite number of independent normal variates. Biometrika **43** (1956), 79–84.

Let S_r, $r=1, 2, \cdots, n$, be the partial sums of n independent standard normal deviates. Let $U_n=\max_r \{S_r\}$. Annis and Lloyd [Biometrika **40** (1953), 35–42; MR **14**, 1102] gave an expression for the expected value of U_n. The author [ibid. **42** (1955), 96–101; MR **16**, 940] gave an expression for the variance of U_n. In the present paper a recurrence relation is given from which one could compute the higher moments of U_n. *M. Muller.*

Cohen, E. Richard. Standard errors of the residues in a least-squares analysis. Phys. Rev. (2) **101** (1956), 1641–1642.

"The covariance matrix between input and output data of a least-squares analysis is computed and hence the standard error of a residue may be evaluated. This leads directly to a means of evaluating the consistency of the data more precisely than is usually the case." (From the author's summary.) *B. Epstein* (Stanford, Calif.).

Elfving, G. An expansion principle for distribution functions with application to Student's statistic. Ann. Acad. Sci. Fenn. Ser. A. I. no. **204** (1955), 8 pp.

Let X have a continuous d.f. and $\{Y_i\}$ be independent of X with $Y_i\to 0$ in probability as $i\to\infty$. Let t be measurable with $t(x, y)\to x$ as $y\to 0$. By expanding $P\{t(X, y)\leq t\}$ as a sequence of functions $\sum F_j(t, y)$ in an expeditious way and integrating with respect to the d.f. of Y_i, one obtains an asymptotic series for the d.f. of $t(X, Y_i)$ as $i\to\infty$. In the application of the title a formula with two terms (using the normal d.f.) gives accuracy to order n^{-2}. *J. Kiefer.*

Edgett, George L. Multiple regression with missing observations among the independent variables. J. Amer. Statist. Assoc. **51** (1956), 122–131.

With reference to Matthai [Sankhyā **11** (1951), 145–152; MR **13**, 571] the author finds explicit expressions for the maximum-likelihood estimators in the case of a trivariate normal population. The possibility that the missing observations imply a selective bias is not discussed

H. Wold (Uppsala).

Harley, B. I. Some properties of an angular transformation for the correlation coefficient. Biometrika **43** (1956), 219–224.

Good, I. J. Rational decisions. J. Roy. Statist. Soc. Ser. B. **14** (1952), 107–114.

This paper deals first with the relationship between the theory of probability and the theory of rational behaviour. A method is then suggested for encouraging people to make accurate probability estimates, a connection with the theory of information being mentioned. Finally, Wald's theory of statistical decision functions is summarised and generalised and its relation to the theory of rational behaviour is discussed. (Author's summary.)

L. J. Savage (Chicago, Ill.).

Haldane, J. B. S.; and Smith, Sheila Maynard. The sampling distribution of a maximum-likelihood estimate. Biometrika **43** (1956), 96–103.

With sufficient regularity conditions the ith cumulant, k_i, of maximum-likelihood estimators, based on a sample of N, are of the form

$$k_i=\sum_{j=i-1}^{\infty} G_{ij}N^{-j}.$$

The authors obtain expressions for G_{10}, G_{11}, G_{21}, G_{22}, G_{32}, and G_{43}. A conjecture is made of a necessary and sufficient condition for $G_{1i}=0$, $i\geq 1$, in the case of discrete distributions and a sufficient condition for $G_{11}=0$ is found. The uses of these results are discussed.

I. R. Savage (Stanford, Calif.).

Jung, Jan. On linear estimates defined by a continuous weight function. Ark. Mat. **3** (1956), 199–209.

Let X be a random variable having c.d.f. $F((x-\mu)/\sigma)$, where $F(y)$ is a known c.d.f. and where μ and σ are unknown parameters. Consider the order statistics $x_{(1)}\leq x_{(2)}\leq\cdots\leq x_{(n)}$, obtained by rearranging n independent observations on X. Lloyd has shown that one can determine constants $h_\nu^{(n)}$ so that linear estimates of μ or σ of the form $\theta^*=\sum_{\nu=1}^{n} h_\nu^{(n)}x_{(\nu)}$ are unbiased and of minimum variance. For some specific distributions explicit values of $h_\nu^{(n)}$ are obtained by Downton, Sarhan and others. There is, however, no numerical solution is general. In this paper it is shown that if one makes $h_\nu^{(n)}=n^{-1}h(\nu(n+1)^{-1})$, where $h(u)$ is a suitably chosen continuous function depending only on $F(y)$, then the corresponding estimate $\theta^*(h)$ is consistent and asymptotically efficient. The choice of a "best" weighting function $h(u)$ depends on solving an associated minimization problem.

B. Epstein (Stanford, Calif.).

Williams, R. M. The variance of the mean of systematic samples. Biometrika 43 (1956), 137–148.

Godambe, V. P. A unified theory of sampling from finite populations. J. Roy. Statist. Soc. Ser. B. 17 (1955), 269–278.

To estimate the sum T of the N elements of a finite population from a sequence of n random draws (made with or without replacement) the author introduces the most general type of linear estimator of T, viz.

$$(1) \qquad e_s = \sum_{\lambda \in s} \beta_{s\lambda} x_\lambda,$$

where $\beta_{s\lambda}$ is defined in advance for all logically possible sequences s and for all $\lambda \in s$. (1) includes the usual types of linear estimators. It is shown that an unbiased linear estimator with least variance does not exist uniquely for the entire class (1). A slightly modified criterion for a best estimator, the least expected variance which takes into account certain prior information, is therefore proposed. Some sampling designs are analyzed accordingly.

H. A. David (Melbourne).

Claringbold, P. J. Matrices in quantal analysis. Biometrics 11 (1955), 481–501.

"In this paper practical methods of relating the binomial variable to the coordinates of experimental designs are given, and matrix methods are employed so that the results may immediately be applied to any experiment with known design matrix.

A considerable time has been devoted in the past to methods supposed to give quick estimation of parameters in quantal analysis. These graphical or semi-graphical methods are usually employed in order to avoid efficient but tedious probit analysis in routine work. In this paper it will be shown that quick efficient solution is afforded by use of the angular transformation". (From the author's introduction.) *D. M. Sandelius.*

Good, I. J.; and Toulmin, G. H. The number of new species, and the increase in population coverage, when a sample is increased. Biometrika 43 (1956), 45–63.

The authors present here a further development of the theory expounded by Good [Biometrika 40 (1953), 237–264; MR 15, 809]. Suppose that a random sample of size N is drawn from a multinomial population. Methods are given for estimating, knowing only the contents of this sample, the number of classes which will be represented r times in a second sample of size λN. These also enable us to estimate the number of different classes and the proportion of the whole population represented in the second sample. A formula is found for the variance of the estimate. The methods are applied to samples of words from "Our mutual friend", to an entomological sample, and to a sample of nouns from Macaulay's essay on Bacon. The authors state a lemma, determination of a set of numbers whose factorial moments are specified, which is presumably well-known, although they cannot give a reference. One such reference is a paper by the reviewer [Ann. Math. Statist. 18 (1947), 574–581; MR 9, 234]. *S. Kullback* (Washington, D.C.).

van Eeden, Constance. Maximum likelihood estimation of ordered probabilities. II. Math. Centrum Amsterdam. Statist. Afdeling Rep. S 196 (VP7) (1956), 12 pp.

This is a continuation of an earlier paper [van Eeden, same Rep. S 188 (VP5) (1956); MR 17, 640].

B. Epstein (Stanford, Calif.).

Whittle, P. Some distribution and moment formulae for the Markov chain. J. Roy. Statist. Soc. Ser. B. 17 (1955), 235–242.

Let (p_{ij}) be a Markov transition matrix for a system with a states, having 1 as a simple characteristic root and the only root of magnitude 1. Let n_{jk} be the number of direct transitions from j to k in $N+1$ consecutive positions. It was shown by Bartlett [Proc. Cambridge Philos. Soc. 47 (1951), 86–95; MR 12, 512] that $\hat{p}_{jk}=n_{jk}/\sum_k n_{jk}$ is the maximum likelihood estimate of p_{ik}. The author obtains an explicit expression for the joint distribution of the n_{jk} and uses it to (1) obtain an asymptotic expression for the likelihood function when the n_{jk} are large, (2) show asymptotic normality of the n_{jk}, (3) obtain their asymptotic covariances (4) show the asymptotic chi-square distribution (a^2-a d.f.) of the goodness-of-fit criterion proposed by Bartlett, loc. cit., (5) obtain the moments of certain sample functions, leading to approximations to the means and covariances of the $\hat{p}_{jk}$. The author attributes (2), (4), and parts of (5) to Bartlett, loc. cit., and (3) is related to work of Patankar [ibid. 51 (1955), 96–102; MR 16, 494]. *T. E. Harris.*

Kullback, S. An application of information theory to multivariate analysis. II. Ann. Math. Statist. 27 (1956), 122–146.

Let $I(1:2)$ denote one of the Kullback-Leibler numbers for discriminating between probability densities f_1 and f_2 [same Ann. 22 (1951), 79–86; MR 12, 623]. The author proposes testing the null hypothesis that $f_2 \in F$, a set of probability densities, by means of the test criterion I^* which is obtained from $I(1:2)$ by replacing the parameters in f_1 by unbiased estimators which are functions of the sufficient statistics, and taking the minimum of the resulting expression as f_2 ranges through F. It is shown that, provided only unbiased estimators are used, this procedure is equivalent to maximising the information. The method is applied to four multivariate problems: (a) the hypothesis that a k-variate normal population has the covariance matrix σ; (b) the hypothesis of equality of r means for each of k variates for r k-variate normal populations with different covariance matrices, and with the same covariance matrix; (c) the multivariate linear hypothesis, including the case of a subhypothesis; (d) the hypothesis of equality of the covariance matrices of r k-variate normal populations. The asymptotic distributions of I^* are discussed in detail for the four problems. The reviewer is unable to understand the need for unbiased estimators in connexion with information-theory concepts. *D. V. Lindley* (Cambridge, England).

Barton, D. E.; and David, F. N. Tests for randomness of points on a line. Biometrika 43 (1956), 104–112.

Assume that on a line of unit length $(n-1)$ points are distributed at random. Let the points be dropped at distances x_i from one end ($i=1, 2, \cdots, (n-1)$) with $x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1}$. Let $d_1=x_1$, $d_i=x_i-x_{i-1}$ $(1<i\leqq(n-1))$, $d_n=1-x_{n-1}$. Let $\{g_i\}$ denote the set of d_i's ranked according to their magnitude. The positional rank of the interval g_i (which is the ith largest in size) is r_i. Let $G=\sum_{i=1}^{n} r_i g_i$ and $G^*=n\sum_{i=1}^{n} d_i g_i$. Approximations to the moments of G are given. The authors mention they plan in a later note to study the power functions of tests of randomness based on G and G^* respectively.

M. Muller (New York, N.Y.).

Guiraum Martin, Alfonso. Modification of Lawley's test for a small sample. Las Ciencias **18** (1953), 753–756. (Spanish)

David, H. A. On the application to statistics of an elementary theorem in probability. Biometrika **43** (1956), 85–91.

Let $P_{ij\cdots k}$ denote the joint probability of r ($\leq n$) events $A_i, A_j, \cdots, A_k$, say. Let S_r be the sum of the $\binom{n}{r}P$'s with different subscripts. Then the probability $p_{m,n}$ of the realization of m events out of n is given by

$$(*) \qquad p_{m,n}=\sum_{l=0}^{n-m}(-1)^l\binom{m+l-1}{m-1}S_{m+l}$$

[see, e.g., W. Feller, An introduction to probability theory and its applications, Wiley, New York, 1950; MR 12, 424]. Among the statistical applications of this formula are the distribution of the largest to the sum of a group of independent sample variances [Cochran, Ann. Eugenics **11** (1941), 47–52; MR 3, 171] and various extreme deviate distributions [e.g. Halperin, Greenhouse, Cornfield, and Zalokar, J. Amer. Statist. Assoc. **50** (1955), 185–195; MR **16**, 726]. Tables are given for some of the tests discussed. Certain simple approximations to (*) are shown to be adequate in many cases of interest.

P. Meier (Baltimore, Md.).

Terpstra, T. J. A generalization of Kendall's rank correlation statistic. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 690–696; **59** (1956), 59–66= Indag. Math. **17** (1955), 690–696; **18** (1956), 59–66.

M. G. Kendall's rank correlation statistic S is generalized to cases where the number of series of observations being ranked is more than two, and where the number of observations in each cell within a series may differ from one. The test statistic reduces to Kendall's S when one has two ranked series of equal length with one observation per cell. Some asymptotic distribution results are given.

B. Epstein (Stanford, Calif.).

Castoldi, Luigi. Identità generalizzata di Wald nella teoria probabilistica delle sequenze casuali. Boll. Un. Mat. Ital. (3) **11** (1956), 22–27.

This paper falls into the trap of identifying $E(X_j)|_{j=N}$ with $E(X_N|N)$, where N is a random integer. In consequence, the announced generalization must be interpreted either as an uninteresting identity or as an interesting, but false, generalization of Wald's "fundamental identity of sequential analysis." *L. J. Savage.*

Linnik, Yu. V. On polynomial statistics in connection with the analytic theory of differential equations. Vestnik Leningrad. Univ. **11** (1956), no. 1, 35–48. (Russian)

Let $\xi=(X_1, X_2, \cdots, X_n)$ be a random vector with independently and identically distributed components. Let $P(\xi)$ be a polynomial statistic of degree $m\geq 1$ and assume that all similar terms are already collected in $P(\xi)$. Such a polynomial statistic is called admissible if it contains X_i^m. Denote by $P_0(\xi)$ the homogeneous polynomial containing all the terms $X_1^{k_1}X_2^{k_2}\cdots X_n^{k_n}$ (with their respective coefficients) where $k_1+k_2+\cdots+k_n=m$. Replace in each term of $P_0(\xi)$ each factor X_j by $m(m-1)\cdots(m-k_j+1)$. In this manner one obtains a new polynomial $\pi_0(m)$. The polynomial $P(\xi)$ is called an ordinary polynomial if $\pi_0(m)\neq 0$ for all integer m. The author proves the following theorem: Assume that: (i) $P(\xi)$ is an admissible and ordinary polynomial which depends only upon the differences of the components of the vector ξ; (ii) for some γ such that $0<\gamma<1$ the expectation $E|X_j|^\gamma$ exists; (iii) the probability $\Pr[P(\xi)=\text{const}]<1$; (iv) the statistic $P(\xi)$ is stochastically independent of the statistic $T=X_1+X_2+\cdots+X_n$. Then the population is normal.

The proof of the theorem is carried by first showing that the characteristic function of the population distribution function is an entire function without zeros and by studying the growth of this function and the differential equation satisfied by the cumulant generating function. The paper presents a new approach to the problem of characterizing the normal population by the independence of translation-invariant statistics from the sample mean. As in all earlier attempts, such as the reviewer's [C. R. Acad. Sci. Paris **238** (1954) 444–445; MR **15**, 542] or V. Paskevich's [Prikl. Mat. Meh. **17** (1953), 513–516; MR **17**, 53], probably too restrictive assumptions are made. Nevertheless the paper marks a serious progress in the solution of the above-mentioned characterization problem. However, the diversity of the attempts to solve this problem seems to indicate that a final solution, not restricted to polynomial statistics, is still to be found.

E. Lukacs (Washington, D.C.).

Teicher, Henry. On the multivariate Poisson distribution. Skand. Aktuarietidskr. **37** (1954), 1–9.

Let $\{\alpha_n\}$ be a sequence of independent identically distributed m-dimensional random vectors whose coordinate take only the values zero and unity. Let N be a Poisson random variable independent of the $\{\alpha_n\}$. Let $X=\sum_{n=1}^{N}\alpha_n$. The vector X is said to have a multivariate Poisson distribution. Let $\{x_1, x_2, \cdots, x_m\}$ be the coordinates of X. The author derives the frequency function of X by means of passages to the limit in the frequency functions of the corresponding multinomial distribution and gives various recurrence relations for the computation of these frequency functions.

The generating function $g(z_1, z_2, \cdots, z_m)=E(\prod z_j^{x_j})$ is also derived and it is shown that for g to be the generating function of a m-variate Poisson it is necessary and sufficient that, for every $j=1, 2, \cdots, m$ and every positive c, $g(z_1, \cdots, z_{j-1}, c, z_{j+1}, \cdots, z_m)$ be proportional to the generating function of a $(m-1)$-variate Poisson. It is also shown that if X is m-variate Poisson and X is the sum of two independent random vectors Y and Z then both Y and Z are multivariate Poisson. Finally, the author shows directly that if $EN\to\infty$ then the distribution of $[X-E(X)][E(N)]^{-\frac{1}{2}}$ tends to a multivariate normal distribution. *L. Le Cam* (Berkeley, Calif.).

McFadden, J. A. An approximation for the symmetric, quadrivariate normal integral. Biometrika **43** (1956), 206–207.

Pillai, K. C. S. On the distribution of the largest or the smallest root of a matrix in multivariate analysis. Biometrika **43** (1956), 122–127.

The author derives an approximation for obtaining the upper (5% or less) percentage points for the largest root and the lower (5% or less) percentage points for the smallest root of a matrix in multivariate analysis for any number of roots up to 5. The 5% and 1% points for two roots for certain values of parameters involved have been tabulated. For two roots the distributions of the largest and smallest roots are also developed as hypergeometric series. *S. Kullback* (Washington, D.C.).

Lawley, D. N. A statistical examination of the centroid method. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954–55), 175–189 (1956).

A modified centroid method used in factor analysis is considered in which (i) the residual variances are assumed to be known, (ii) the sign changes are determined in advance and (iii) the process is applied to the covariance matrix. The author gives various large-sample results which yield a method of statistical estimation and some tests of significance of residuals. The modified centroid method avoids in some respects the real difficulties met with in practice, by the restriction placed by the assumptions (i) and (ii). The main justification for the assumption (i), according to the author, is that the centroid method can be compared with a maximum-likelihood method in which also the residual variances are assumed to be known. It is shown that the modified centroid method has in general fairly high efficiency and has the advantage of being relatively simple to perform, as compared with the maximum-likelihood method. However, it has the disadvantage that the χ^2-test of significance is not so convenient and exists as yet only in a large-sample form. *T. Kitagawa* (Fukuoka).

Kempthorne, O. A class of experimental designs using blocks of two plots. Ann. Math. Statist. **24** (1953), 76–84.

Let n and r denote the number of treatments and replicates, respectively. For the design considered treatment i should be placed in a block together with each of the treatments $i+s, i+s+1, \cdots, i+s+r-1$ $(i+1, \cdots, n)$, where each of the numbers $i+s, i+s+1, \cdots, i+s+r-1$ is to be reduced modulo n. This structure is possible only if $2s+r-1=n$. Formulas are obtained for estimates of treatment comparisons without and with utilization of inter-block information. *D. M. Sandelius.*

Nair, K. R. Relation between efficiency of incomplete block designs and the intra-class correlations associated with incomplete and complete blocks. J. Indian Soc. Agric. Statistics **4** (1952), 149–152.

The author shows that the relation mentioned in the title exists mathematically, and he concludes that it is unnecessary and undesirable to establish this relation statistically, as has been done earlier. *D. M. Sandelius.*

Lomnicki, Z. A.; and Zaremba, S. K. Some applications of zero-one processes. J. Roy. Statist. Soc. Ser. B. **17** (1955), 243–255.

The quantity Ux is defined to be unity if the real variable x is positive, and zero otherwise. From any discrete-parameter stationary process $\{x_t\}$ a process $\{Ux_t\}$ can be derived. The autocovariances of a series Ux_t are very easily calculated, and it is shown that from this series and modifications of it inferences can be made concerning the autocorrelations, frequency of upcrosses, etc., of the original process $\{x_t\}$. *P. Whittle.*

Lur'e, A. L. On construction of a mathematical theory of statistical control of production processes. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 2, 113–119. (Russian)

Let D be the distance of the upper and lower control limits from the center line of a control chart, n the number of items inspected after each time interval t. The author considers the problem of finding optimum values for D, n, and t, on the basis of minimizing the cost of operating the statistical quality control procedure and the loss due to defective production. *G. E. Noether* (Boston, Mass.).

Èĭdel'nant, M. I. The simplest nonparametric method of quality control. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 52–62. (Russian)

General discussion of single sampling plan with zero defectives allowed in sample. Relevant formulas are derived and tables are provided. *G. E. Noether.*

Romanovskiĭ, V. I. Duality theorems for the hypergeometric distribution. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 22–28. (Russian)

Given N items, M of which belong to a certain category (defectives in a lot submitted for acceptance). In a sample of size n, m are found to belong to this category. The author finds the aposteriori probability distribution of M if the apriori distribution is given by a) $P(M)=1/(N+1)$, b) $P(M)=C_N^M p^M(1-p)^{N-M}$, where p is given. Consider the acceptance sampling procedure which accepts the lot if the number of defectives d in a sample of size n is $\leq$ some number c, and rejects the lot if $d>c$. Assuming that the apriori distribution of M is given by b), where p is the average proportion of defectives for the production process, the author finds expressions for certain probabilities of the type $P(M\leq Np \mid d\leq c, n)$. An example is given. *G. E. Noether* (Boston, Mass.).

Diveev, R. H. Sorting into kinds. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **11** (1953), 69–73. (Russian)

Lots are said to belong to one of three classes on the basis of the proportion of defective items included. The following sampling scheme is discussed: A lot is classified as belonging to the first, second, or third class depending on whether the number m of defective items in a sample of size n is $\leq c_1$, $c_1<m\leq c_2$, $c_2<m$, where c_1 and c_2 are suitably chosen numbers. Upper limits for the various probabilities of misclassification are given. *G. E. Noether* (Boston, Mass.).

Èĭdel'nant, M. I. Effectiveness of a form of statistical control. Vestnik Leningrad. Univ. **11** (1956), no. 1, 74–88. (Russian)

A classification of acceptance sampling plans is given and the effectiveness of certain special plans on the basis of a loss function is evaluated. *G. E. Noether.*

Linn, Hans-Jochen; und Pöschl, Klaus. Eine einfache Deutung des komplexen Korrelationskoeffizienten. Arch. Elek. Übertr. **10** (1956), 105–106.

Miller, Leslie H. Table of percentage points of Kolmogorov statistics. J. Amer. Statist. Assoc. **51** (1956), 111–121.

Let $S_n(x)$ be the empiric distribution function based on a sample of size n from the completely specified distribution function $F(x)$. Let $D_n=\max\{S_n(x)-F(x)\}$ and $D_n^*=\max|S_n(x)-F(x)|$. More extensive tables of the percentage points in the distribution of D_n and D_n^* than have been published previously are given. *M. Muller.*

Nomachi, Yukio. Auxiliary tables for the applications of n-dimensional t-distributions to certain class of empirical functions. Bull. Math. Statist. **6** (1955), no. 1-2, 25–47.

See also: Kitagawa and Mitome, p. 934; Hartley, Shrikande and Taylor, p. 934; Mendelsohn, p. 935; Zia-ud-Din, p. 936; Roy and Sarhan, p. 936; Mallows, p. 950; Wooding, p. 978; Gihman, p. 980.

Theory of Games, Mathematical Economics

Newell, G. F. Mathematical models for freely-flowing highway traffic. J. Operations Res. Soc. Amer. 3 (1955), 176–186.

An attempt is made to construct models for the motion of cars on a highway in a manner analogous to the way one treats the motion of molecules in a gas as described by the kinetic theory of gases. In kinetic theory, little is assumed concerning the details of the interaction between molecules, and yet the macroscopic model obtained is very satisfactory. From this, we are led to expect that even a very crude model of the interaction between individual cars will give a quite reasonable description of the collective behavior of large groups of cars. Some simple estimates of the dependence of average velocity on density and distribution of cars in various lanes of a multilane highway based upon very crude models seem to confirm this expectation. *C. C. Torrance.*

Schützenberger, M. P. A tentative classification of goal-seeking behaviours. J. Mental Sci. **100** (1954), 97–102.

An attempt is made here "to show how certain of the more complex goal-seeking behaviors, seen in both machines and men, can be described, and the principles made clear, by a uniform mathematical framework of ideas". This framework is essentially the obtaining of an optimization by a suitable succession of sub-optimizations that may be characterized by "span of foresight" and "flexibility". Some simple heuristic examples are given.

C. C. Torrance (Monterey, Calif.).

Uzawa, Hirofumi. Note on Leontief's dynamic input-output system. Proc. Japan Acad. **32** (1956), 79–82.

The system in question can be reduced to the form $x=C\hat{x}_+$, where C is an $n\times n$ matrix of nonnegative elements and the components of $\hat{x}_+$ are max $(\hat{x}_i, 0)$. The author assumes C to be nonsingular. For x^0 to be a possible initial position, x^0 must be in the cone spanned by the columns of C. C is defined to be "retrograde" if certain inequalities involving the principal minors of its inverse have nonnegative solutions. This property is shown to be connected with the existence of solutions of the above system for any admissible x^0. *R. Solow.*

Abrams, I. J. A note on the optimal character of the (s, S) policy in the inventory problem. Univ. California Publ. Statist. **2** (1956), 185–194.

"In a recent paper [Econometrica **21** (1953), 586–596; MR **15**, 333], A. Dvoretzky, J. Kiefer and J. Wolfowitz considered an inventory problem in which the penalty for failing to meet the demand is a constant. They showed that in such a situation there does not always exist an optimum policy of the so-called (s, S) type.

In the present paper we consider the same inventory problem but assume that the penalty for not meeting the demand depends in a particular way on the excess of the demand over the available stock. It is shown that for loss functions of a certain type the optimum policy is always of the (s, S) type. The numbers s and S which determine this optimal policy depend on the distribution of demand. When the distribution of demand is known, the constants can be obtained by a procedure described in section 2. If, however, the distribution of demand is not known, the constants s and S have to be estimated from observations. In section 3 of the present paper estimates of s and S are described. A confidence interval for S is given, and under some restrictive assumptions the estimate of S is shown to be asymptotically normally distributed. Both estimates of s and S are shown to be strongly consistent." (Author's introduction.)

J. Wolfowitz (Ithaca, N.Y.).

Arrow, Kenneth J.; and Debreu, Gerard. Existence of an equilibrium for a competitive economy. Econometrica **22** (1954), 265–290.

Two theorems concerning the existence of a competitive equilibrium for certain economic systems are proved. The economic models are integrated models with fairly general assumptions concerning production and consumption.

S. Malmquist (Stockholm).

Gale, David. The law of supply and demand. Math. Scand. **3** (1955), 155–169.

The following model of economic equilibrium is considered: Let U_i $(i=1, \cdots, m)$ be economic units, each of which supplies or consumes a vector x of commodities in a set X_i, assumed compact, convex and containing the null vector. A feasible operation is a set of vectors x_i, where $\sum_{i=1}^{m} x_i \geqq 0$, $x_i \in X_i$ for all i. For any price vector p, there is a set-valued supply function $S_i(p)$, assumed to be a convex set, where $p \cdot x \geqq 0$ for all $x \in S_i(p)$ and S_i is continuous in an appropriate sense. The basic theorem states that there exists a price vector p_0 and a feasible operation x_i $(i=1, \cdots, m)$ such that $x_i \in S(p_0)$ for each i. This theorem is closely related, as pointed by the author, to the equilibrium theorems proved by Arrow and Debreu [see the paper reviewed above], and also, though not mentioned by the author, to that of McKenzie [Econometrica **22** (1954), 147–161]. As in these papers, the proof is essentially based on a fixed-point theorem.

Mathematically, the novelty is basing the proof on the following lemma: Let S be a bounded continuous set-valued function from the unit simplex P in Euclidean n-space such that $S(p)$ is non-empty and convex for all p and, if $x \in S(p)$, then $x \cdot p \geqq 0$; then there exists $p \in P$ and $x \in S(p)$ such that $x \geqq 0$. This lemma is derived from a result of Knaster, Kuratowski, and Mazurkiewicz; it appears to provide the most elementary topological approach to equilibrium theorems in economics.

An elementary result on uniqueness is also obtained. Some other special results on Leontief models, the relation between supply functions and preference orderings, and the relation between competitive equilibrium and optimal distribution are derived; all the results and the proofs are essentially contained in previously published work.

K. J. Arrow (Stanford, Calif.).

Majumdar, Tapas. Choice and revealed preference. Econometrica **24** (1956), 71–73.

Vidale, Marcello L. A graphical solution of the transportation problem. Operations Res. **4** (1956), 193–203.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Jerison, Meyer. An algebra associated with a compact group. Pacific J. Math. 5 (1955), 933–939.

Let G be a compact abelian group, R a commutative complex B-algebra. Let $R(G)$ be the B-algebra of continuous maps $x: G \to R$, with multiplication à la convolution and max norm. Main theorem (§ 3): Let G_1, G_2, R_1, R_2 be as above, R_1 and R_2 further restricted to possess identities but no other idempotents. Then $R_1(G_1)$ is isometrically isomorphic to $R_2(G_2)$ ($R_1(G_1) \approx R_2(G_2)$) if and only if $R_1 \approx R_2$ and $G_1 \cong G_2$. The author hoped to solve a problem in topology due to Fox: namely, let X be the circle (i.e. rim), let Y and Z be compact spaces; does homeomorphism of $X \times Y$ and $X \times Z$ imply that of Y and Z? The approach fails, and the author has some worthwhile remarks as to why this is so. *J. G. Wendel.*

Lie Groups, Lie Algebras

Freudenthal, Hans. The existence of a vector of weight 0 in irreducible Lie groups without centre. Proc. Amer. Math. Soc. 7 (1956), 175–176.

The author give an affirmative answer to a conjecture which, according to the author, was made by R. Bott and H. Samelson. The conjecture is: Every irreducible linear semi-simple compact Lie group with trivial center has a vector of weight 0. The author proves this by showing that every weight is a linear combination of root forms if and only if the group is irreducible and has a 0 weight. From this it immediately follows that every weight takes values $= 0 \pmod{2\pi i}$ at an ad h for h an element in the Cartan subgroup if and only if each root form takes values $= 0 \pmod{2\pi i}$ at that ad h. It is stated that Harish-Chandra has given a proof different from the above proof. *H. Yamabe* (Minneapolis, Minn.).

Hunt, G. A. A theorem of Elie Cartan. Proc. Amer. Math. Soc. 7 (1956), 307–308.

The author gives a proof for a well-known theorem that any two maximal abelian subgroups of a compact connected Lie group G are conjugate to each other within G. The proof depends on a lemma: For given two elements x and y in the Lie algebra $\mathfrak{G}$ of G, there exists an inner automorphism A_σ of $\mathfrak{G}$ so that $[x, A_\sigma y] = 0$. This σ is obtained as a minimum point in G of a non-positive function $(x, A_\sigma y)$ on G, where (u, v) stands for Cartan's fundamental bilinear form. Once this is established, the theorem is proved by applying the lemma to the generating elements of given two maximal abelian subgroups which are toroidal groups. *H. Yamabe* (Minneapolis, Minn.).

Rozenfel'd, B. A. Geometrical interpretation of compact simple Lie groups of class E. Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 600–603. (Russian)

The author announces a geometrical interpretation of the compact exceptional Lie groups E_6, E_7, E_8. To this end the author constructs "elliptic planes" over the algebras $K \otimes O$, $Q \otimes O$, $O \otimes O$, where, K, Q, O denote the algebras of complex numbers, quaternions, Cayley numbers. The main result is that E_6, E_7, E_8 may be interpreted as groups of motions of the elliptic planes over $K \otimes O$, $Q \otimes O$ and $O \otimes O$. *W. T. van Est* (Utrecht).

Mušic, N. P. On maximal reducible subgroups of the simple real Lie groups. Kazan. Gos. Univ. Uč. Zap. 114, no. 2 (1954), 195–204. (Russian)

The author determines the maximal reducible Lie subgroups of the real forms (in their standard linear representations) of the following real simple Lie groups: A_n, B_n, D_n, C_n. *W. T. van Est* (Utrecht).

See also: Nagai, p. 1004.

Topological Vector Spaces

Pavel, Monica. Quelques observations sur l'inversion des applications linéaires et continues des espaces vectoriels topologiques. Com. Acad. R. P. Romîne 5 (1955), 1569–1572. (Romanian. Russian and French summaries)

Rephrasing of well-known results in terms of the author's notion of "linear retraction" (i.e., projection). *V. L. Klee* (Los Angeles, Calif.).

Marinescu, Georges. Espaces polynormés, duals des espaces localement convexes. C. R. Acad. Sci. Paris 241 (1955), 1693–1695.

The author discusses some of the ideas used in dual spaces in terms of polynorms, that is, a directed system of norms (generalized to allow $+\infty$ as a value) satisfying the following conditions. (a*) For each $x \in E$ there is an α such that $\|x\|_\alpha < +\infty$. (b*) If α and β are given, there exists γ such that for every x in E $\|x\|_\gamma \leq \|x\|_\alpha$ and $\|x\|_\gamma \leq \|x\|_\beta$. *M. M. Day* (Urbana, Ill.).

Ionescu Tulcea, C. T. Théorèmes ergodiques. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 3 (1954), no. 4–5, 65–69. (Romanian. Russian and French summaries)

By averaging over increasing compact neighborhoods of 0 and using Eberlein's general ergodic theorem [Trans. Amer. Math. Soc. 67 (1949), 217–240; MR 12, 112] the author gives a mean ergodic theorem for weakly continuous unitary representations $t \to U_t$ of locally compact abelian groups. The condition (K) should read: For each $t \in G$ $\lim_P \varphi_t = 1$. *M. M. Day* (Urbana, Ill.).

Kinokuniya, Yoshio. A synthetic light on the distributions and their stochasticity. Mem. Muroran Univ. Engrg. 1, no. 5 (1954), 821–829.

Banach Spaces, Banach Algebras

Călugăreanu, G. Remarques sur les normes d'un espace vectoriel. Science R. P. Roumaine 1 (1953), 19–22 (1954).

French version of an earlier Romanian paper [Acad. R. P. Române, Bul. Şti. Şecţ. Şti. Mat. Fiz. 4 (1952), 69–73; MR 15, 631].

Del Pasqua, Dario. Su una nozione di varietà lineari disgiunte di uno spazio di Banach. Rend. Mat. e Appl. (5) 13 (1955), 406–422.

Two linear subspaces M and N of a normed linear space are said to be "disjoint" provided there is a positive k such that $\|x+y\| \geq k\|x\|$ whenever $x \in M$ and $y \in N$. The

largest such k (which is the distance between N and the unit sphere of M) is called the "index of disjointness" of M and N. The paper, expository in nature, is devoted to a study of these notions and their relationship to other parts of functional analysis (projections and orthogonality boundedness of linear functionals, etc.). To the list of references should be added Kober, Compositio Math. 7 (1939), 135–140 [MR 1, 59] and Mackey, Trans. Amer. Math. Soc. 57 (1945), 155–207 [MR 6, 274; 7, 620].

V. L. Klee, Jr. (Los Angeles, Calif.).

Makarov, B. M. On the topological equivalence of B-spaces. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 17–18. (Russian)

Theorem 1. Every infinite-dimensional Banach space can be renormed by a complete norm not topologically equivalent to the original norm. Theorem 2. If X_1 and X_2 are Banach spaces, if X_1 is set-theoretically a subset of X_2, and if X_2 admits a total family of bounded linear functionals whose restriction to X_1 are bounded linear functionals on X_1, then X_1 is also algebraically and topologically a subspace of X_2. Various consequences are drawn from Theorem 2. *E. Hewitt* (Seattle, Wash.).

Braunschweiger, Chris C. A geometric construction of L-spaces. Duke Math. J. **23** (1956), 271–280.

The author describes a method of constructing from some linear or Banach spaces an L-space and gives conditions under which the new space is isomorphic to the original space. *M. M. Day* (Urbana, Ill.).

Rogosinski, W. W. Continuous linear functionals on subspaces of $\mathfrak{L}^p$ and $\mathfrak{C}$. Proc. London Math. Soc. (3) **6** (1956), 175–190.

A continuous linear functional B on a closed linear subspace $\mathfrak{G}$ of one of the (complex) Banach spaces $\mathfrak{L}^p$ ($1\leq p<\infty$) or $\mathfrak{C}$ may, as is well known, be extended to the entire space and then represented as an integral with respect to a measure (kernel). The present paper investigates the nature of such kernels which are minimal in the sense that their norms in the conjugate Banach spaces are equal to those of B on $\mathfrak{G}$, with a view to applications in extremum problems in analysis. The principal line of this investigation is through the study of the existence and nature of a maximal function $G\in\mathfrak{G}$ for B, that is a function of unit norm such that $B(G)$ is equal to the norm of B on $\mathfrak{G}$. From the relation between such maximal functions (when they exist) and minimal kernels, information about the latter is obtained.

For the space $\mathfrak{L}^p$ ($1<p<\infty$) a unique maximal function always exists, and the minimal kernel is unique. In the case of $\mathfrak{L}^1$ or $\mathfrak{C}$ the absence in general of the existence and uniqueness of a maximal function and of the uniqueness of a minimal kernel complicates the situation. Here those functionals B which possess a maximal function are characterized and properties of maximal functions are obtained.

Finally, applications to moment problems are obtained by use of the results in the case where $\mathfrak{G}$ is the linear subspace generated by a given linearly independent set of functions. *W. R. Transue* (Gambier, Ohio).

Hanner, Olof. On the uniform convexity of L^p and l^p. Ark. Mat. **3** (1956), 239–244.

Clarkson [Trans. Amer. Math. Soc. **40** (1936), 396–414] defined uniformly convex normed linear spaces and showed that the modulus of convexity in the spaces l^p and L^p was at least $\delta'(\varepsilon)=1-(1-(\frac{1}{2}\varepsilon)^r)^{1/r}$, where $r=p$ if $p\geq 2$ and $r=p/(p-1)$ if $1<p<2$. In this paper the author shows that the first of these estimates is largest possible and that for $1<p<2$ the largest possible modulus $\delta=\delta(\varepsilon)$ is found by solving for δ the equation

$$(1-\delta+\tfrac{1}{2}\varepsilon)^p+|1-\delta-\tfrac{1}{2}\varepsilon|^p=2.$$

M. M. Day (Urbana, Ill.).

Luxemburg, W. A. J.; and Zaanen, A. C. Some remarks on Banach function spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 110–119.

This paper represents a continuation of studies begun in Luxemberg's thesis [Tech. Hogesch. Delft, 1955; MR **17**, 285]. A measure μ on a space Δ (totally σ-finite) and a function $\varrho(f)$, f a measurable non-negative function on Δ, are given. ϱ has the essential character of a norm, and $X=\{f \mid f \text{ measurable on } \Delta,\ \varrho(|f|)<\infty\}$. Suitable and natural axioms for ϱ insure $X\neq\emptyset$ and X is a Banach space, $\|f\|_X=\varrho(|f|)$. If g is measurable

$$\|f\|_X=\sup\{\textstyle\int|fg|d\mu \mid \|f\|_X\leq 1\},\ \|g\|_{X''}=\sup\{\int|fg|d\mu \mid \|f\|_{X'}\leq 1\},$$

X' and X'' consist of functions having finite $\|\cdot\|_{X'}$, resp. $\|\cdot\|_{X''}$. There is a constant γ such that if $0\leq f_n\uparrow f$, then $\gamma\|f\|_X\leq\lim\|f_n\|_X$. Furthermore, $\gamma\|f\|_X\leq\|f\|_{X''}\leq\|f\|_X$. Furthermore, $\|f\|_{X''}=\inf\{\lim\|f_n\|_X \mid f_n\uparrow|f|\}$. If $\sigma(X,X')$ is the analog of the classical weak topology, then the unit sphere of X is dense (in this topology) in the unit sphere of X''. If $\Delta_n\uparrow\Delta$, $\mu(\Delta_n)<\infty$, let $X^b=\{f \mid f=0$ outside some Δ_n, f bounded$\}$. Let $X^x=\{f \mid \|f\|_X$ is absolutely continuous$\}$. Then $X^x=\bigcap X^b$ (intersection over all families $\Delta_n\uparrow\Delta$, $\mu(\Delta_n)<\infty$). $(X^x)^*=X'$ if and only if $X^x=X^b$.

B. Gelbaum (Minneapolis, Minn.).

Marinescu, G. Sur la formule de Lagrange dans les espaces de Banach. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **3** (1954), no. 4–5, 71–72. (Romanian. Russian and French summaries)

This note follows earlier work of Vaĭnberg [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 3(49), 139–143; MR **14**, 55] but considers functions from one normed space E to a conjugate space F. *M. M. Day* (Urbana, Ill.).

Dunford, Nelson; and Schwartz, J. T. Convergence almost everywhere of operator averages. J. Rational Mech. Anal. **5** (1956), 129–178.

The deduction of Birkhoff's pointwise ergodic theorem from von Neumann's mean ergodic theorem was given first by N. Wiener [Duke Math. J. **5** (1939), 1–18]. The reviewer observed [Proc. Imp. Acad. Tokyo **16** (1940), 280–284; MR **2**, 105] that for a general operator T in a Lebesgue space $L_1(S)$ the pointwise ergodic theorem can be derived from the mean ergodic theorem for T by applying a general principle of Banach provided one can show that for each $f\in L_1(S)$ the inequality

$$\sup_n |n^{-1}\sum_{k=0}^{n-1}(T^k f)(s)|<\infty \tag{1}$$

holds for almost all s. N. Dunford obtained jointly with D. S. Miller [Trans. Amer. Math. Soc. **60** (1946), 538–549; MR **8**, 280], a general class of operators T, including those induced by equi-measure transformations, for which (1) and the mean ergodic theorem both hold. E. Hopf [J. Rational Mech. Anal. **3** (1954), 13–45; MR **15**, 636] obtained a more general class of operators T satisfying (1) and comprising the Markoff operators. The result covers the earlier Doob-Kakutani [Doob, Trans. Amer. Math.

Soc. **44** (1938), 87–150; Kakutani. Proc. Imp. Acad. Tokyo **16** (1940), 49–54; MR **1**, 343] pointwise ergodic theorem for Markoff processes and for bounded measurable functions f.

The present paper gives a decisive result in the direction of applying the general principle of Banach. Thus it is proved that (1) holds good for any operator T in $L_1(S)$ provided one has (2) $\|T\|_{L_1} \leq 1$ and ess. $\sup_s |(Tf)(s)| \leq$ ess. $\sup_s |f(s)|$. The crucial point in the proof is a deep lemma due to E. Hopf (loc. cit.) assuring the dominated ergodic theorem: Let P be a positive operator in L_1 with $\|P\|_{L_1} \leq 1$. Then, for every real $f \in L_1(S)$ and n, we have $\int_E f(s)ds \geq 0$, where

$$E=\{s; \sup_{m \leq n} m^{-1} \sum_{k=0}^{m-1} (P^k f)(s) \geq 0\}.$$

The authors give a simplified and somewhat generalized proof of the lemma, and a full application of it to deduce the pointwise ergodic theorem from the mean ergodic theorem. *K. Yosida* (Tokyo).

Dinculeanu, Nicolae. Sur les théorèmes ergodiques. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **7** (1955), 887–901. (Romanian. Russian and French summaries)

Let E be a space with measure m and let G be a semigroup of transformations T of E into itself such that each T^{-1} preserves measurability and measure zero. For a semigroup generated by a single such T and for each $f \in L$, the space of m-summable functions on E, define $U_n f(x)=n^{-1}\sum_{t \leq n} f(T^t x)$ for all $x \in E$. Consider the conditions: (N) For each $f \in L$ $\lim_n U_n f$ exists in the norm of L. (B) For each $f \in L$ $\lim_n U_n f(x)$ exists for almost all x in E. (DM) There exists K such that

$$V_n(A)=n^{-1}\sum_{t \leq n} m(T^{-t}A) \leq Km(A)$$

for every measurable $A \subset E$. (H) There exists K such that $\limsup_n V_n(A) \leq Km(A)$ for every measurable $A \subset E$.

Dunford and Miller [Trans. Amer. Math. Soc. **60** (1946), 538–549; MR **8**, 280] showed (N) equivalent to (DM) and (N) implies (B). In this paper it is shown that (B) is equivalent to (H) and that all these results generalize to abelian semigroups with n generators or n parameters. *M. M. Day* (Urbana, Ill.).

Jacobs, Konrad. Ergodentheorie und fastperiodische Funktionen auf Halbgruppen. Math. Z. **64** (1956), 298–338.

In an earlier paper [Math. Z. **61** (1955), 408–428; MR **16**, 719] the author discussed the decomposition of a uniformly convex normed space H into a direct sum of the subspaces P and F, where P is the set of vectors in H almost periodic under a given bounded group G of linear operators in H, and where F is the set of furtive vectors under G. Here he wishes to extend this conclusion to reflexive spaces and is able to prove it for bounded abelian semigroups with identity, using the definition of almost periodic vector of a semigroup of operators due to Maak [Acta Math. **87** (1952), 33–57; MR **13**, 910]. The author observes that his technique surely applies to solvable semi-groups; the reviewer feels that it will also work out for amenable bounded semigroups, i.e., semigroups with both right- and left-invariant means on the bounded functions. *M. M. Day* (Urbana, Ill.).

Reuter, G. E. H. A note on contraction semigroups. Math. Scand. **3** (1955), 275–280 (1956).

This paper is concerned with strongly continuous semigroups consisting of either positive contraction or transition operators on an abstract (L)-space. Given two such semi-groups $\Sigma=\{P_t\}$ and $\Sigma'=\{P'\}$, Σ' is said to dominate Σ if $P_t' x \geq P_t x$ for all $x \geq \theta$ and $t \geq 0$. It is shown that if Σ is a transition semi-group, then no distinct contraction semi-group Σ' dominates Σ. On the other hand, if Σ is merely a contraction semi-group and not a transition semi-group, then there exist infinitely many distinct Σ' which dominate Σ. In fact, if Σ is generated by Ω and c belongs to the (L)-space, $c \geq \theta$ with $\|c\| \leq 1$, then $\Omega_c x=\Omega x-(u, \Omega x)c$ defined on the domain of Ω generates a semi-group Σ_c which dominates Σ; here the functional $(u, x)=\|x^+\|-\|x^-\|$. Further Σ_c is a transition semi-group if and only if $\|c\|=1$. *R. S. Phillips.*

Miyadera, Isao. A note on strongly ergodic semi-group of operators. Kōdai Math. Sem. Rep. **7** (1955), 55–57.

The author is concerned with semi-groups of linear bounded operators $[T(\xi)]$ on a B-space X, strongly continuous for $\xi>0$ with $\int_0^1 \|T(\xi)x\|d\xi<\infty$ for each $x \in X$. A semi-group of this sort which is strongly Abel [or $(C-\alpha)$] summable to the identity at $\xi=0$ is said to be of class $(0, A)$ [or $(0, C_\alpha)$]. The principal result is the following theorem, which was proved for the case $\alpha=1$ by the reviewer [Ann. of Math. (2) **59** (1954), 325–356; MR **15**, 718]: A necessary and sufficient condition that a semi-group of class $(0, A)$ be of class $(0, C_\alpha)$ is that there exist an $\omega \geq$ such that

$$\sup_{\lambda>0,\, k} \left\| \frac{\alpha \cdot (k!)}{\Gamma(k+\alpha+1)} \sum_{i=0}^{k} \frac{\Gamma(k+\alpha-i)}{(k-1)!} [R(\lambda+\omega; A)]^{i+1} \right\| <\infty;$$

here $R(\lambda; A)$ is the resolvent of the complete infinitesimal generator A. The above theorem together with one of the author's earlier results [Tôhoku Math. J. (2) **6** (1954), 231–242; MR **16**, 1031] on the generation of semi-groups of class $(0, A)$ provide a necessary and sufficient condition that a closed linear operator be the complete infinitesimal generator of a semi-group of class $(0, C_\alpha)$. *R. S. Phillips* (Los Angeles, Calif.).

Bonsall, F. F. Dual extremum problems in the theory of functions. J. London Math. Soc. **31** (1956), 105–110.

The author has reexamined a recent paper (R–S) by W. W. Rogosinski and H. S. Shapiro [Acta Math. **90** (1953), 287–318; MR **15**, 516] from the viewpoint of linear-space theory and has simplified and strengthened some of their results. Let V be a normed linear space, and V^* its dual space. If G is a subspace of V, let $G^\perp$ be the subspace of V^* orthogonal to G. The chief tool used in (R–S) was the following simple consequence of the Hahn-Banach theorem: if $\phi \in V^*$, then $\|\phi\|_{G^\perp}=\text{dist}\{\phi \text{ to } G^\perp\}$, moreover, the extremum on the right is achieved. By recourse to compactness arguments, (R–S) showed that in certain cases, the left-hand extremal was also achieved. In this paper, the author uses an alternative approach. Theorem 2: Let $x_0 \in V$, and regard it as a member of V^{**}. Then, $\|x_0\|_{G^\perp}=\text{dist}\{x_0 \text{ to } G\}$ and the left hand extremal is achieved. For the applications given in (R–S), let H^p be the subspace of L^p consisting of the functions f such that $\int_0^{2\pi} f(t)e^{int}dt=0$ $(n=1, 2, \cdots)$ and let K^p be the subspace of H^p determined by $\int_0^{2\pi} f(t)dt=0$. When $1<p<\infty$, choose $V=L^p$, $G=H^p$, and have $V^*=L^q$, $G^\perp=K^q$ and $V^{**}=V$, $G^{\perp\perp}=G$. The general theorems at once yield the corollary that for any $g \in L^q$, $p^{-1}+q^{-1}=1$,

$$\max_{f \in H^p,\ \|f\| \leq 1} \left| \int_0^{2\pi} f(t)g(t)dt \right| = \min_{k \in K^q} \|g+k\|_q.$$

When $p=1$, choose V as $C[0, 2\pi]$ and G as the linear span of the functions $\{e^{int}\}$ $(n=0, 1, \cdots)$. Then, it follows that for any function g which is continuous on $[0, 2\pi]$, or more generally, which belongs to $C+K^\infty$,

$$\max_{f\in H^1,\ \|f\|\leq 1}\left|\int_0^{2\pi} f(t)g(t)dt\right| = \min_{k\in K^\infty}\|g+k\|_\infty.$$

In (R-S), this last result was established only for a more restricted class of functions g. In a final section, the author formulates another general question: if Γ is a subspace of V^*, then when it is true that for any $x_0 \in V$, $\|x_0\|=\operatorname{dist}\{x_0 \text{ to } \Gamma_\perp\}$; here, $\Gamma_\perp$ is the subspace of V orthogonal to Γ in the natural pairing of V^* with V. Several illustrations of this situation are given.

R. C. Buck (Madison, Wis.).

Cristescu, Romulus. Opérateurs-produit. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 4 (1955), no. 8, 23–33. (Romanian. Russian and French summaries).

Dans cette note on considère les opérations $P(x, y)$ définies sur $X\times Y$ à valeurs dans Z, où X, Y, Z sont des structures vectorielles et qui sont biadditivies et à valeurs positives lorsque les deux arguments sont positifs. On donne les propriétés générales de ces opérations et l'on indique leurs applications aux équations fonctionnelles. (From the author's summary.) *E. Hewitt.*

Cvid, F. A. On the completion of normed K-spaces. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 37–44. (Russian)

Let X be a real normed linear space and also a vector lattice in which every bounded set has a least upper bound. It is shown that if X has certain special properties connecting its norm and its partial ordering, then the norm-completion of X has certain other such properties.

E. Hewitt (Seattle, Wash.).

Vulih, B. Z. Characteristic properties of the product in linear partially ordered spaces. Leningrad. Gos. Ped. Inst. Uč. Zap. 89 (1953), 3–8. (Russian)

This note contains a much simplified version of the author's previous construction of multiplication in certain K-spaces [Mat. Sb. N.S. 22(64) (1948), 27–78; MR 10, 46].

E. Hewitt (Seattle, Wash.).

See also: Albrycht, p. 953; Jerison, p. 986; Suzuki, p. 989; Misonou, p. 990; Michael, p. 990; Aquaro, p. 992.

Hilbert Space

Fage, M. K. On the parabolic basis of V. Ya. Kozlov. Ukrain. Mat. Ž. 4 (1952), 212–214. (Russian)

Let $\mathfrak{E}$ be a Hilbert space, $\{\mathfrak{H}_n\}_1^\infty$ a "subspace" basis in $\mathfrak{E}$, i.e., each $x\in\mathfrak{E}$ can be written uniquely in the form $x=\sum_1^\infty x_n$, $x_n\in\mathfrak{H}_n$. Let $\mathfrak{G}_n$ be the "adjoints" of the $\mathfrak{H}_n$ $(\mathfrak{H}_n\perp\mathfrak{G}_m, n\neq m)$, $I^{(n)}=\sum_1^n I_k$ where I_k is the (not necessarily self-adjoint) projection on $\mathfrak{H}_k$, $\mathfrak{H}^{(n)}=\sum_1^n\mathfrak{H}_k$, $\mathfrak{G}^{(n)}=\sum_1^n\mathfrak{G}_k$, $\mathfrak{F}_n=G^{(n)}\mathfrak{H}_n$ where $G^{(n)}$ is the projection $\mathfrak{G}^{(n)}$. Then $\mathfrak{H}_n=I^{(n)}\mathfrak{F}_n$ and we set $\mathfrak{F}_n'=I^{(n)*}\mathfrak{H}_n$, $P_n=(E-I^{(n)*})\mathfrak{H}_n$ (E the identity). Theorem: $P_n\perp\mathfrak{H}^{(n)}$. If the $\mathfrak{F}_n'$ are pairwise orthogonal, $\{\mathfrak{H}_n\}_1^\infty$ is called a Kozlov basis. Theorem: If $\{\mathfrak{H}_n\}_1^\infty$ is a Kozlov basis, then the P_n are pairwise orthogonal. Finally, a Kozlov basis is a "rectifiable basis, i.e., can be converted by an analog of the Gram-Schmidt process to an orthogonal "subspace" basis. Otherwise stated, the "idempotent" resolution of the identity $\{I_n\}$ $(I^{(n)}\to E$ in strong topology and $I_nI_m=0$ if $n\neq m)$ is similar to an orthogonal resolution of the identity $(I_n=SJ_nS^{-1}$, S a constant bounded operator with bounded inverse S^{-1} and $\sum_1^n J_k\to E$ in the strong topology, $J_k^2=J_k=J_k^*$, $J_kJ_l=0$, $k\neq l)$. *B. Gelbaum.*

Kilpi, Yrjö. Zur Theorie des mehrdimensionalen Momentenproblems. Ann. Acad. Sci. Fenn. Ser. A. I. no. 205 (1955), 13 pp.

Let $R=\{\lambda\}$ be n-dimensional space, $\varrho(\lambda)=\varrho(\lambda_1, \cdots, \lambda_n)$ a distribution function defined in R, $\{P(\lambda)\}$ the polynomials of the form

$$P(\lambda_1, \cdots, \lambda_n)=\sum_{\gamma_1,\cdots,\gamma_n} x_{\gamma_1}\cdots x_{\gamma_n}\lambda_1^{\gamma_1}\cdots\lambda_n^{\gamma_n},$$

$\mathfrak{H}$ the closure of the set $\{P(\lambda)\}$ in the Lebesgue space $L^2(d\varrho)$ and considered as an Hilbert space, H_ν $(\nu=1, \cdots, n)$ the operators in $\mathfrak{H}$ defined by $H_\nu P(\lambda_1, \cdots, \lambda_n)=\lambda_\nu P(\lambda)$, M_ν the subspace of $\mathfrak{H}$ spanned by the elements 1, $H_\nu 1$, $H_\nu^2 1$, $\cdots$, and $H_{\nu M}$ the restriction of H_ν to M_ν. Then: 1) $H_\nu+iH_\mu$ $(\mu, \nu=1, \cdots, n)$, is a normal operator in the sense of Nakano. 2) If the operators $H_{\nu M}$ are self-adjoint in M_ν, then the H_ν are self-adjoint in $\mathfrak{H}$. 3) If the n-dimensional moment problem

$$(*)\qquad \mu_{\gamma_1\cdots\gamma_n}=\int_R \lambda_1^{\gamma_1}\cdots\lambda_n^{\gamma_n}d\varrho(\lambda)$$

has a solution, and if the 1-dimensional moment-problems $\mu_{0\cdots\gamma_\nu\cdots 0}=\int_R \lambda_\nu^{\gamma_\nu}d\varrho(\lambda)$ are determined $(\nu=1, \cdots, n)$, then the problem (*) is determined. [Reviewer's note: Closely related results were obtained, among others, and by a different method, by A. Devinatz, Trans. Amer. Math. Soc. 77 (1954), 455–480, especially Th. 4; MR 16, 584.]

M. Cotlar (Mendoza).

David, Fernando Soares. On commutativity of operators with continuous spectra. Gaz. Mat., Lisboa 12 (1951), no. 50, 103–105. (Portuguese)

Let A and B be Hermitian operators with continuous spectra. The author gives a necessary and sufficient condition for A and B to commute (i.e. for their resolutions of the identity to commute) in terms of the commutativity of certain associated operators with pure point spectra. These associated operators are particular kinds of sums related to the Stieltjes-integral representations of A and B by their resolutions of the identity. This commutativity condition is given an interpretation in connection with the compatibility of observables in quantum mechanics. Then follows an application to the case of position and momentum for linear motion, and as a consequence the author enunciates an uncertainty principle concerning simultaneous measurement of these two observables. *A. E. Taylor* (Los Angeles, Calif.).

Suzuki, Noboru. On automorphisms of W^*-algebras leaving the center elementwise invariant. Tôhoku Math. J. (2) 7 (1955), 186–191.

Given a w^* algebra $\mathfrak{M}$ and a $*$ automorphism α which leaves the center of $\mathfrak{M}$ elementwise invariant, when is it true that α is inner (i.e. that there is some unitary U in $\mathfrak{M}$ such that $A^\alpha=UAU^*$ for all A in $\mathfrak{M}$)? This paper gives various sufficient conditions for α to be inner which are new for type II and III algebras, and which contain the result of Kaplansky that when $\mathfrak{M}$ is type I then α is always inner. Unfortunately, for type II and III algebras, the class of inner automorphisms satisfying the author's conditions is rather small. *J. Feldman.*

Misonou, Yosinao. Generalized approximately finite W^*-algebras. Tôhoku Math. J. (2) **7** (1955), 192–205.

A W^* algebra of type II_1 is called approximately finite if it is generated by a family of mutually commuting W^* subalgebras of type I, each containing the center of the the original algebra. This generalizes one of von Neumann's definitions of an approximately finite factor of type II_1 on a separable Hilbert space. In case the II_1 is a factor, and none of the generating type I subalgebras are the trivial subalgebra (i.e. multiples of the identity operator), the cardinality of the subfamily determines the II_1 up to algebraic isomorphism, and vice versa. Any approximately finite W^* algebra can be written as a C^* sum of "uniform" parts, each such part being the tensor product of a commutative W^* algebra and an approximately finite factor. These results were also obtained by H. Widom [Thesis, Univ. of Chicago, 1955].

J. Feldman (Princeton, N.J.).

See also: Potapov, p. 958.

TOPOLOGY

Smirnov, Yu. On the theory of completely regular spaces. Moskov. Gos. Univ. Uč. Zap. **155**, Mat. **5** (1952), 137–155. (Russian)

Let R be a T_1-space with closure operator $^-$. Let Φ be a closed subset of R, and $O\Phi$ an open neighborhood of Φ. A system Σ of pairs $(\Phi, O\Phi)$ is called dense if for every $(\Phi, O\Phi) \in \Sigma$ there exists a neighborhood $O^*\Phi$ of Φ such that $(O^*\Phi)^- \subset O\Phi$, $(\Phi, O^*\Phi) \in \Sigma$, and $((O^*\Phi)^-, O\Phi) \in \Sigma$. The largest dense system, denoted by Σ_R, is called the regularity system of R. The author first proves that two closed sets A, B are functionally separated in R if and only if $(A, B') \in \Sigma_R$. (The construction differs hardly at all from Urysohn's original construction of separating continuous functions in normal spaces, which is, indeed, utilized in all such constructions.) It follows that R is completely regular if and only if all pairs $(\{p\}, U(p))$ are in Σ_R, where p is an arbitrary point of R and $U(p)$ is an arbitrary neighborhood of p.

The author next takes up normal spaces. Theorem: R is normal if and only if: 1) every closed G_δ set in R is the set where some continuous real-valued function vanishes; 2) for every closed set $\Phi \subset R$, and every neighborhood $O\Phi$, there is a closed G_δ, Φ_0, such that $\Phi \subset \Phi_0 \subset O\Phi$. Examples show that neither 1) nor 2) is sufficient for normality. Theorem: Let R be normal, Φ a closed subset of R and f a continuous real-valued function defined on Φ, $0 \leq f \leq 1$. Then f admits a continuous extension f_0 over R such that $f_0 = 0$ exactly where $f = 0$, if and only if $\{x : x \in \Phi, f(x) = 0\}$ is a G_δ in R.

The author also gives a detailed description of a space N constructed first by Nemyckiĭ. The points of N are all pairs of real numbers (x, y) with $y \geqq 0$. For $y > 0$, (x, y) has ordinary Euclidean neighborhoods. For points $(x, 0)$, a generic neighborhood consists of $(x, 0)$ plus the interior of a circle tangent to $(x, 0)$. It is shown among many other things that N is completely regular and non-normal. [See also the reviewer's analysis of a similar space, Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126.] Finally the author shows that Tihonov's construction of a regular non-completely regular space [Math. Ann. **102** (1929), 544–561] can be carried out starting with any completely regular, non-normal space. In surveying the status of such examples, he is apparently unaware of the reviewer's example of a regular space admitting only constant real-valued continuous functions [Ann. of Math. (2) **47** (1946), 503–509; MR **8**, 165]. A summary of the paper under review appeared earlier [Dokl. Akad. Nauk SSSR (N.S.) **62** (1948), 749–752; MR **10**, 315].

E. Hewitt (Seattle, Wash.).

Smirnov, Yu. M. Remark on the work, "On a problem connected with the metrizability of topological spaces." Ukrain. Mat. Ž. **4** (1952), 220–223. (Russian)

Theorem: Let R be a countably compact Hausdorff space that is the union of 2 complementary closed sets each of which has a countable basis. Then R has a countable basis. The proof given is a correction of an earlier one [Ukrain. Mat. Ž. **3** (1951), 161–163; MR **14**, 893].

E. Hewitt (Seattle, Wash.).

Heider, L. J. Directed limits on rings of continuous functions. Duke Math. J. **23** (1956), 293–296.

If X^D is a discrete space and (X, T) is a completely regular space with the same underlying point set X, then the identity map $i: X^D \to (X, T)$ is continuous, and by a theorem of M. H. Stone [Trans. Amer. Math. Soc. **41** (1937), 375–481, th. 88], i has a continuous extension which sends βX^D, the Stone-Čech compactification of X^D, onto $\beta(X, T)$. Thus, any completely regular space (X, T) can be realized as a subspace of a decomposition of βX^D. The mapping i also induces a natural isomorphism of the ring $C(X, T)$ of real continuous functions on (X, T) into the ring $C(X^D)$. Using this isomorphism, theorems of Gelfand and Kolmogoroff [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **22** (1939), 11–15] and Hewitt [Trans. Amer. Math. Soc. **64** (1948), 45–99; MR **10**, 126] relating the maximal ideals of $C(X, T)$ to the points of $\beta(X, T)$ are reformulated as relations between the maximal ideals of $C(X, T)$ and the points of βX^D.

M. Jerison.

Michael, Ernest. Continuous selections. I. Ann. of Math. (2) **63** (1956), 361–382.

A function ϕ on a topological space X into the family 2^Y of all non-empty subsets of a topological space Y is called a carrier and will be denoted by $\phi: X \to 2^Y$. If $A \subset X$, the restriction of ϕ to A is denoted by $\phi|A$. A carrier ϕ is called lower semi-continuous (l.s.c.) if for every open subset V of Y, $\{x \in X : \phi(x) \cap V \text{ is non-empty}\}$ is open in X. A selection for ϕ is a continuous $f: X \to Y$ such that $f(x) \in \phi(x)$ for every $x \in X$.

The author is primarily concerned with the following problem. What restrictions must be placed on X and $\mathcal{S} \subset 2^Y$ in order that for every closed subset A of X and every l.s.c. $\phi: X \to \mathcal{S}$, every selection for $\phi|A$ can be extended to a selection for ϕ? (The author shows that the restriction to l.s.c. carriers involves very little loss of generality.) It is shown that if $\mathcal{S}$ contains all one-point subsets of Y then the above holds if and only if every l.s.c. carrier $\phi: X \to \mathcal{S}$ admits a selection, in which case for purposes of this review, we shall call X a selection space for Y mod $\mathcal{S}$. It is shown that if $\mathcal{S}$ contains all one-point subsets of Y and contains Y, then if X is a selection space for Y mod $\mathcal{S}$, then for every closed subset A of X, every continuous $f: A \to Y$ has a continuous extension over X. An example is given to show that the converse is false.

If Y is a Banach space, let $\mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed and convex}\}$, and let $\mathcal{C}(Y) = \{S \in \mathcal{F}(Y) : S \text{ is compact or } S = Y\}$. Let R denote the real line.

The following characterization theorems for T_1-spaces are obtained: (1) X is normal if and only if X is a selection space mod $C(R)$ if and only if X is a selection space mod $C(Y)$ for every separable Banach space Y. (2) X is collectionwise normal if and only if X is a selection space mod $C(Y)$ for every Banach space Y. (3) X is normal and countably paracompact if and only if X is a selection space mod $\mathcal{F}(R)$ if and only if X is a selection space mod $\mathcal{F}(Y)$ for every separable Banach space Y. (4) X is paracompact if and only if X is a selection space mod $\mathcal{F}(Y)$ for every Banach space Y. A similar, but more complicated characterization of perfectly normal spaces is obtained. Examples are given to show that some of these theorems cannot be improved.

Next some generalizations of a theorem of Bartle and Graves [Trans. Amer. Math. Soc. **72** (1952), 400–413; MR **13**, 951] on the existence of a selection for the inverse mapping of a linear transformation between Banach spaces are obtained.

A collection $\mathcal{B}$ of closed subsets of a topological space X is said to dominate X if, whenever a subset A of X has a closed intersection with every member of a subcollection $\mathcal{B}_1$ of $\mathcal{B}$ that covers A, then A is closed. (Note that a closed locally finite covering of X dominates X.) It is shown that X is paracompact (normal, perfectly normal, normal and countably paracompact) if and only if X is dominated by a collection of closed paracompact (normal, perfectly normal, normal and countably paracompact) subsets. [Part of this result was obtained independently by K. Morita, Proc. Japan Acad. **30** (1954), 711–717; MR **16**, 1041.] [Reviewer's remark: In Theorem 3.1''' (c), $\mathcal{H}(Y)$ should be replaced by $\mathcal{D}(Y)$.] *M. Henriksen.*

Dugundji, James; and Michael, Ernest. On local and uniformly local topological properties. Proc. Amer. Math. Soc. **7** (1956), 304–307.

An ordering α of the subsets of a space X is defined to be proper provided: (a) if $W\alpha V$, then $W\subset V$; (b) if $W\subset V$ and $V\alpha R$, then $W\alpha R$; if $W\alpha V$ and $V\subset R$, then $W\alpha R$. For example, $W\alpha V$ might mean (i) that every pair of points in W is contained in a connected subset of V, or (ii) that W is contractible in V, etc. X is locally of type α if, whenever $x\in X$ and V is a neighborhood of x, then there is a neighborhood W of x with $W\alpha V$. In (i) above, this is equivalent to local connectedness of X, and in (ii) to local contractibility. X is uniformly locally of type α if to every $\varepsilon>0$ there is a $\delta>0$ with $S_\delta(x)\alpha S_\varepsilon(x)$ for every $x\in X$. It is proved that if α is a proper ordering on the subsets of a metrizable space X, and if X is locally of type α, then there is a metric ϱ on X, agreeing with the topology, such that (X, ϱ) is uniformly locally of type α. The above definitions are generalized to coverings of a space X; the analogous theorem holds also in this more general case. *E. E. Floyd* (Charlottesville, Va.).

★**Lima, Élon Lages. Topologia dos espaços métricos. [Topology of metric spaces.]** Notas de Matemática No. 10. Instituto de Matemática Pura e Aplicada do Conselho Nacional de Pesquisas, Rio de Janeiro, 1954. xii+117 pp. (mimeographed).

Première partie du cours de topologie générale donné par l'auteur dans le cadre de son enseignement de géométrie supérieure durant l'année 1954 à l'intention des étudiants en mathématiques de 2ème et 3ème année à la faculté de Philosophie de Rio de Janeiro. Cet exposé, clair et précis, émaillé d'exemples et d'exercices, suit d'assez près Nicolas Bourbaki et peut constituer une excellente introduction au traité de topologie générale de ce dernier. C'est, du reste, une tendance qui semble se dessiner dans divers pays. Cet appel aux bons services des espaces métriques, refuges où les appels de l'étudiant à son intuition euclidienne seront encore entendus, après avoir été utilisés par lui comme des étapes préliminaires vers les chemins plus rudes de la topologie générale. La suite des différents paragraphes qui constituent ce cours est définie ainsi: 0) Préliminaires. 1) Espaces métriques. 2) Fonctions continues. 3) Sphères et ensembles ouverts. 4) Fermés. 5) Suites. 6) Ensembles convexes et topologie. 7) Sous espaces. 8) Espace complet. 9) Espace séparable. 10) Compacts. 11) Homéomorphismes. Annexe: Lectures recommandées. *J. Riguet* (Paris).

de Groot, J. On some problems of Borsuk concerning a hyperspace of compact sets. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 95–103.

Borsuk [Fund. Math. **41** (1955), 168–202; MR **16**, 946] has introduced a metrization of the space 2^M of compact subsets of a metric space M as follows: $\varrho_e(X, Y)$ is defined to be the greatest lower bound of all $\varepsilon>0$ such that there exist continuous $\phi: X\to Y$ and $\psi: Y\to X$ with $\varrho(x, \phi(x))\leq\varepsilon$, all $x\in X$, and $\varrho(y, \psi(y))\leq\varepsilon$, all $y\in Y$. In answer to a problem of Borsuk, the author shows that separability of M does not necessarily imply separability of 2^M (M is a 2-cell in the example). However, for a separable metric M, it is shown that the subspace H of 2^M consisting of compact ANR's is separable. Borsuk also introduced another metric into the space 2_h^M of compact ANR's in a separable metric M. In answer to a question of Borsuk, it is shown that 2_h^M is separable. *E. E. Floyd.*

Griffith, H. C. The enclosing of cells in three space by simple closed surfaces. Trans. Amer. Math. Soc. **81** (1956), 25–48.

Let E^k denote the unit k-cube, for $1\leq k\leq 3$, and let C be a topological k-cell in Euclidean 3-space R^3. $\mathfrak{T}=\mathfrak{T}(C)$ denotes the set of all "Cartesian slices" of C. That is, $T\in\mathfrak{T}$ if there is a homeomorphism $h: E^{k-1}\times E^1\to C$ (onto) such that $T=h(E^{k-1}\times x)$ for some $x\in E^1$. Let $\mathfrak{T}_h$ denote the set of "slices" relative to a given h. If $A, B\subset R^3$, and every point of $A-B$ has a closed neighborhood which intersects A in a polyhedron, then A is locally polyhedral modulo B. Definition I. Suppose that for every $T\in\mathfrak{T}$, and every $\varepsilon>0$, there is a topological 2-sphere K, containing T in its interior, and lying in the ε-neighborhood of T, such that $K\cap C$ is either an element of $\mathfrak{T}$ or the union of two disjoint elements of $\mathfrak{T}$, the latter case occurring only when T intersects the interior of C. Then C has property $\mathcal{P}$. If these conditions hold for every $T\in\mathfrak{T}_h$, then C has property $\mathcal{P}$ relative to $\mathfrak{T}_h$. (This generalizes a definition due to Harrold [Duke Math. J. **21** (1954), 615–621; MR **16**, 846]; Harrold's definition was restricted to 1-cells and 1-spheres.) Definition II. Let $T\in\mathfrak{T}$, and let ε be a positive number. Then $\mathfrak{D}(T, \varepsilon)$ is the set of all topological disks $D\subset R^3$ such that (1) $D\cap C\in\mathfrak{T}$, (2) the boundary of D is disjoint from C, (3) D is locally polyhedral modulo C, (4) $\varrho(T, D\cap C)<\varepsilon$, where ϱ is a distance function closely related to the Hausdorff distance, and (5) if $C-D$ has two components C_1, C_2, then there is an $\eta>0$ such that if N is a connected set of diameter $<\eta$, and N meets both C_1 and C_2, then N meets D. If $\mathfrak{D}(T, \varepsilon)$ is non-empty for every $T\in\mathfrak{T}$ (or every $T\in\mathfrak{T}_h$), then C has the disk property (or the disk property relative to $\mathfrak{T}h$.) Theorem. If C has property $\mathcal{P}$ (relative to $\mathfrak{T}_h$), then C has the disk property (relative to $\mathfrak{T}_h$).

Definition III. C has the uniform disk property relative to $\mathfrak{T}_h$ if for every $\omega>0$ there is a $\delta>0$ such that if $T \in \mathfrak{T}_h$ and $\varepsilon>0$, then there is a $D \in \mathfrak{D}(T, \varepsilon)$ such that (1) D lies in the ω-neighborhood of T and (2) the distance from C to the boundary of D is $>\delta$. If C has the uniform disk property relative to every $\mathfrak{T}_h$, then C has the uniform disk property. Definition IV. C has the enclosure property if for every $\varepsilon>0$ there is a polyhedral topological 2-sphere K, such that K contains C in its interior and K lies in the ϵ-neighborhood of C. Theorem. If there is an h such that C has property $\mathcal{P}$ relative to $\mathfrak{T}_h$ and the uniform disk property relative to $\mathfrak{T}_h$, then C has the enclosure property. Definition. Suppose that for each homeomorphism h: $E^{k-1}\times E^1 \to C$ (onto) there are (1) points a, b, lying in $h(E^{k-1}\times 0)$ and $h(E^{k-1}\times 1)$ respectively (lying in the interiors of the latter sets for $k>1$), (2) disjoint topological rays A, B, with initial points a, b, and (3) polyhedral 2-spheres $M_1, M_2, \cdots$, such that (i) $A\cap C=a$, and $B\cap C=b$, (ii) $A\cup B$ is locally polyhedral modulo C, (iii) each M_i lies in the $1/i$-neighborhood of C, (iv) each M_i contains C in its interior, and (v) each M_i meets each of the rays A, B in a single point. Then C has the strong enclosure property. Theorem. If C is a k-cell ($1\leqq k\leqq 3$) and has any two of the following three properties, then it also has the third: (1) property $\mathcal{P}$; (2) the uniform disk property; (3) the strong enclosure property. The proofs are by explicit geometric construction, and are rather long.

E. E. Moise (Ann Arbor, Mich.).

Williams, R. F. Reduction of open mappings. Proc. Amer. Math. Soc. **7** (1956), 312–318.

Let X and Y be compact metric spaces and f be an open mapping of X onto Y. Proofs are given of the following theorems. I. If f is monotone and R is a non-empty open subset of Y such that for each $y\subset R$, $f^{-1}(y)$ is a decomposable continuum, then there exists a closed proper subset X' of X, such that $f|X'$ is monotone onto Y and such that $X'\supset X-f^{-1}(R)$. (Theorem I as stated in the paper omits the condition that f be monotone but this extra condition seems clearly to be both necessary and involved in the proof.) II. If f is light, Y is connected, Y is semi-locally-connected at a point $y_0\subset Y$, and $f^{-1}(y_0)$ is nondegenerate, then there exists a proper sub-continuum X' of X such that $f(X')=Y$. III. If Y is perfect and for each $y\subset Y$, $f^{-1}(y)$ is connected and locally connected, then there exists a closed subset X' of X, such that $f|X'$ is monotone onto Y and such that if X'' is a proper closed subset of X', then $f|X''$ is not onto Y.

R. D. Anderson (Princeton, N.J.).

Vaĭnšteĭn, I. A. On closed mappings. Moskov. Gos. Univ. Uč. Zap. **155**, Mat. **5** (1952), 3–53. (Russian)

The author published, several years ago, two short notes on closed mappings [(1) Doklady Akad. Nauk SSSR (N.S.) **57** (1947), 319–321; (2) ibid. **57** (1947), 431–434; MR **9**, 153] which were responsible for considerable of the recent interest in closed mappings. The present paper gives a unified account of the content of those papers, complete with proofs, as well as some new material. Chapter I, on descriptive properties of closed mappings, is largely a completed account of (1) above. Similarly, Chapters III and IV give a full account of (2) above. Chapter II, on irreducible mappings and correct sets, seems to be largely new. Denote by f a closed mapping of a metric space X onto a metric space Y. A closed subset F of X is called a correct set if $F\subset f^{-1}(y)$ for some y and if the image of any open set about F contains y as an interior point. This definition has been used before by the author [ibid. **67** (1949), 9–12; MR **11**, 121] and is related to the quasi-interiority concept of Whyburn [Mem. Amer. Math. Soc. no. **1** (1950); MR **13**, 764]. A closed subset $B\subset f^{-1}(y)$ is a kernel of the point y if B is correct and no proper closed subset of B is correct. The mapping f is irreducible if f maps no proper closed subset of X onto Y. Then (a) every correct set F contains at least one kernel, (b) there exists a closed subset F of X mapped irreducibly onto Y by f, (c) if Y is complete, the set of all $y \in Y$ for which all points of $f^{-1}(y)$ are correct is a dense G_δ. A point $x \in X$ is called semi-correct if the image of every open set about x contains a non-empty open subset of Y. (d) Suppose $y_0 \in Y$ and that D is a closed subset of $f^{-1}(y_0)$. In order that there exist a closed subset F of X mapped irreducibly onto Y and with $F\cap f^{-1}(y_0)=D$, it is necessary and sufficient that D be correct and that every point of D be a semi-correct point. (e) If Y is complete, in order that f be irreducible it is necessary and sufficient that the set of $x \in X$ with $f^{-1}f(x)=x$ be dense in X; this is a generalization of a result of Whyburn for compact spaces.

E. E. Floyd (Charlottesville, Va.).

Dolcher, Mario. Geometria delle trasformazioni continue. Sul numero delle imagini inverse nelle trasformazioni di un dominio piano pluriconnesso. Ann. Univ. Ferrara. Sez. VII. (N.S.) **4** (1954–1955), 1–7.

Let R be a plane disc with r holes, γ_0 its outer boundary and $\gamma_1, \cdots, \gamma_n$ its inner boundaries. Let Φ be a mapping of R into E_2. Let A' be a component of $E_2-\sum \Phi\gamma_i$ and n_i the index of A' relative to $\Phi\gamma_i$. Let r' be the number of indices which are zero. Assuming that $|n_0|\geqq\sum_{i>0}|n_i|$, it is shown that the number of points in $\Phi^{-1}(P')$, where P' is any set of k points in A', can not be less than

$$(|n_0|-\textstyle\sum_{i>0}|n_i|)\cdot(k-r-r').$$

P. A. Smith (New York, N.Y.).

Aquaro, Giovanni. Sopra talune estensioni del teorema di prolungamento di Urysohn-Tietze. Riv. Mat. Univ. Parma **6** (1955), 111–115.

A pair of spaces (X, E) is a u-pair if every continuous map of a closed subset of X into E is extendable over X. J. Dugundji [Pacific J. Math. **1** (1951), 353–367; MR **13**, 373] showed (X, E) a u-pair if X is metric, R locally convex linear. Subsequently, R. Arens [ibid. **2** (1952), 11–22; MR **14**, 191], using pseudometrics, extended the method to show (X, E) a u-pair if X is paracompact normal, E Banach, and X normal, E separable Banach. The author of this paper gives different proofs of the two Arens results using, instead of pseudometrics, a simple approximation device.

J. Dugundji.

Keldyš, Lyudmila. Example of a monotone irreducible mapping of a three-dimensional cube onto a four-dimensional one. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 957–960. (Russian)

A brief, coherent description is given of the example indicated in the title. As the author points out, Anderson has announced examples [Bull. Amer. Math. Soc. **59** (1953), 559] of a similar and stronger type. However, in an area as complicated and poorly understood as this, the present paper will be valuable as a brief outline of one attack on the problem.

E. E. Floyd.

Dyer, Eldon. Certain transformations which lower dimension. Ann. of Math. (2) **63** (1956), 15–19.

The author gives proofs of the following three theorems.

Theorem I. If f is a mapping of the compact metric space M onto the n-dimensional metric space N and for each non-negative integer k and point y of N, $H_k(f^{-1}(y))=0$ and there is an integer m such that the dimension of $f^{-1}(y)$ does not exceed m for any point y of N, then the dimension of M is one of the numbers $n, n+1, \cdots, n+m$.

Theorem II. If f is an open mapping of the compact metric space M onto the n-dimensional metric space N such that the inverse under f of each point of N is a non-degenerate continuous curve and there is a positive number ε such that no simple closed curve in M of diameter less than ε is mapped by f into a point, then the dimension of M is $n+1$.

Theorem III. If M is a compact metric space which carries an n-cycle γ^n such that no closed proper subset of M carries an n-cycle homologous to γ^n on M, then if f is an open mapping of an open subset O of M into a metric space such that the inverse under f of each point of $f(O)$ is a dendron, f is a homeomorphism.

An interesting special case of Theorem III is that in which M is an n-sphere. In this case Theorem III asserts that there does not exist a continuous collection of non-degenerate dendrons filling up any open subset of Euclidean space. It is known [J. H. Roberts, Duke Math. J. 2 (1936), 10–19] (1) that there does exist an upper semi-continuous (u.s.c.) collection of non-degenerate dendrons (arcs and H-curves) filling up the plane (hence any Euclidean space) but (2) that there does not exist an u.s.c. collection of arcs filling the plane. The question as to whether there exists an u.s.c. collection of arcs filling up Euclidean 3-space (or n-space) remains open.

R. D. Anderson (Princeton, N.J.).

See also: Aleksandrov, p. 931; Vagner, p. 942; Jerison, p. 986; Makarov, p. 987; Perkal, p. 999.

Algebraic Topology

Mardešić, Sibe. Sur l'isomorphie des divers groupes d'homologie dans certains espaces fonctionnels. C. R. Acad. Sci. Paris **242** (1956), 983–984.

The author studies in this note the mapping space R^X, where X is a compactum and R a (separable metric) ANR. Consider R as a subspace of the Hilbert space E^w and hence R^X becomes a subspace of $(E^w)^X$ which is a normed vector space. The following two theorems are announced with indications of proof. Theorem 1. R^X is a neighborhood retract of a convex set of a real normed vector space with a uniformly continuous retraction of some ε-neighborhood of R^X. Theorem 2. Let A be a neighborhood retract of a convex set of a real normed vector space with a uniformly continuous retraction of some ε-neighborhood of A. Then, among the various homology groups of A with coefficients in a given abelian group G, the following isomorphisms hold

$$H_V^p(A, G)\approx H_{CV}^p(A, G)\approx H_S^p(A, G)\approx H_C^p(A, G),$$

where the subscripts V, CV, S, C denote that the corresponding homology group is based on the Vietoris convergent cycles, the Vietoris compact convergent cycles, the singular cycles, and the continuous cycles in the sense of Alexandroff-Hopf respectively.

The statements given in this note are inaccurate. The corrected ones given above are indicated in a communication of the author.

The reviewer recalls the following closely related well-known results. Firstly, R^X is an ANR. Secondly, every bounded metric space S is isometric with a closed subset of a convex set of the Banach space $C(S)$ of all bounded continuous real functions on S. [See M. Wojdysławski, Fund. Math. **32** (1939), 184–192.] *S. T. Hu.*

Mardešić, Sibe. Sur l'homologie de l'espace fonctionnel S_m^X et la structure homologique de X. C. R. Acad. Sci. Paris **242** (1956), 1112–1114.

K. Borsuk [Fund. Math. **39** (1952), 25–37; MR **15**, 51] has raised the following problem: If X is a compactum of dimension $k<\infty$ and S_m the m-sphere $k\leq m$, does the homology structure of the mapping space S_m^X depend only on that of X? In the present note, the author answers this question affirmatively for the dimensions p with $0\leq p\leq m-k$. More precisely, for any abelian coefficient group G, the homology group $H_p(S_m^X; G)$, $0\leq p\leq m-k$, depends only on the integral cohomology group $H^k(X)$. Here, homology means either of those homology theories mentioned in the note reviewed above. This result follows from the following theorem: Let X and X' be compacta of dimension k and g: $H^k(X)\to H^k(X')$ a given homomorphism; then g induces a homomorphism

$$\Phi(g): H_{m-k}(S_m^X; G)\to H_{m-k}(S_m^{X'}; G)$$

satisfying the relations

$$\Phi(g'g)=\Phi(g')\Phi(g)\ \ (\Phi(1)=1),$$

where 1 denotes the identity automorphism.

According to a communication of the author, the notation XY in Theorem 3 and thereafter means the Cartesian product $X\times Y$ of the spaces X and Y. *S. T. Hu.*

Hirsch, Guy. L'anneau de cohomologie d'un espace fibré en sphères. C. R. Acad. Sci. Paris **241** (1955), 1021–1023.

Let $p: E\to B$ be an orientable sphere bundle where the fibre is a sphere of dimension $k-1$ and the base space B is compact. In this note the author considers the question of determining the products in the (integral) cohomology ring of E in terms of the cohomology ring of B and certain invariants of the given fibration. Let $p^*: H^*(B)\to H^*(E)$ be the homomorphism induced by the projection p, and let $\mu: H^*(B)\to H^*(B)$, $\psi: H^*(E)\to H^*(B)$ be the homomorphisms of the Gysin sequence of the given sphere bundle [this is the notation used by R. Thom, Ann. Sci. Ecole Norm. Sup. (3) **49** (1952), 109–182; MR **14**, 1004]. If $u\in H^p(B)$ and $v\in H^q(B)$ satisfy the conditions $\mu(u)=0$, $\mu(v)=0$, then $\psi[(\psi^{-1}u)\cdot(\psi^{-1}v)]$ is a non-empty subset of the cohomology group $H^r(B)$, where $r=p+q+k-1$. An easy calculation shows that this subset is a coset of the subgroup $[H^{p+k-1}(B)\cdot v+u\cdot H^{q+k-1}(B)]$. Obviously, knowledge of this coset as a function of u and v would furnish valuable information for the determination of the cohomology ring, $H^*(E)$. The author's main result asserts that if k is even and the integers are used as coefficients for cohomology, then

$$\psi[\psi^{-1}u)\cdot(\psi^{-1}v)]=(-1)^q u\cdot v\cdot W^{k-1}-(-1)^p\langle u, W^k, v\rangle,$$

where $W^k\in H^k(B)$ is the characteristic class of the given bundle, $W^{k-1}\in H^{k-1}(B)$ is a generalized characteristic class [see R. Thom, loc. cit.], and $\langle u, W^k, v\rangle$ is the "triple product" introduced by the reviewer [Bull. Amer. Math. Soc. **57** (1951), 74; for further details, see a forthcoming paper by H. Uehara and W. S. Massey in the symposium

volume, Algebraic geometry and topology, Princeton, 1956]. In case k is odd, this formula is still valid, provided the ring of integers modulo 2 is used as coefficient domain.

W. S. Massey (Providence, R.I.).

Dold, Albrecht; et Thom, René. Une généralisation de la notion d'espace fibré. Application aux produits symétriques infinis. C. R. Acad. Sci. Paris **242** (1956), 1680–1682.

Generalizing the classical notion of a fiber space, the author introduces quasi fiber spaces as follows. A space E is said to be a quasi fiber space over a Hausdorff space B relative to a map p of E onto B if the following condition is satisfied: For each $x \in B$, there exists a fundamental family of open neighborhoods U such that, for each $y \in U$ and $z \in p^{-1}(y)$, we have

$$p_*: \pi_i(p^{-1}(U), p^{-1}(y), z) \approx \pi_i(U, y) \quad (i \geqq 0).$$

For quasi fiber spaces, the usual covering homotopy theorem is false in general. However, it can be replaced by a slightly weaker theorem stated as follows: If $f: P \to E$ is a given map of a polyhedron P and $F: P \times I \to B$ is a homotopy of the composition pf, then F has an arbitrarily small (in B) deformation relative to $P \times 0$ which has a covering homotopy. As a consequence of this theorem, the usual exact homotopy of fiber spaces holds for quasi fiber spaces.

As an interesting application, the author deduces that the qth homotopy group of the infinite symetric product $PS(X)$ of a polyhedron X is isomorphic with the integral homology group $H_q(X)$. By means of this isomorphism, the Hurewicz homomorphism $h: \pi_q(X) \to H_q(X)$ reduces to the homomorphism induced by a natural imbedding $X \subset PS(X)$.

Since $PS(X)$ is an abelian topological semi-group, one may define for each p a quotient space $PS(X)/p$. It follows that the qth homotopy group of $PS(X)/p$ is isomorphic to the homology group $H_q(X; Z_p)$. Two more applications are given at the end of the note.

S. T. Hu.

Milnor, John. Construction of universal bundles. I. Ann. of Math. (2) **63** (1956), 272–284.

Any arc-connected space is the base of a contractible fibre space. The author proves that any given countable connected simplicial CW complex B is also the base of an ∞-universal fiber bundle, the bundle space E and group G being countable CW complexes (and the group operations in G being cellular maps). The construction of E and G is quite explicit. This result is used to give an axiomatic characterization of the homotopy groups in the category of countable CW complexes, using a suitable fibre-bundle condition instead of excision. In the last section, it is shown that any principal bundle over B is induced by a continuous homomorphism of G.

J. Dugundji (Los Angeles, Calif.).

Mizuno, Katuhiko. On homotopy classification and extension. J. Inst. Polytech. Osaka City Univ. Ser. A. **6** (1955), 55–69.

In an earlier paper [same J. **5** (1954), 41–51; MR **16**, 506], the author introduced the complex $T = K(\pi, n, \pi', q, k)$ as a generalization of the Eilenberg-MacLane complex. It is a complex such that $\pi_n = \pi$, $\pi_q = \pi'$ and all other homotopy groups vanish, and $k = k_n^{q+1} \in Z^{q+1}(\pi, n; \pi')$. Then T is determined up to isomorphism by the cohomology class $\mathbf{k}$ of k, which is the Eilenberg-MacLane-Postnikov invariant for spaces Y such that $\pi_i(Y) = 0$ $(0 \leqq i < n, n < i < q)$. We may write π_n, π_q for π, π' and π_i for $\pi_i(Y)$. Suppose in addition that $\pi_i(Y) = 0$ $(q < i < r < 2q-1)$. Then we may also associate with Y (in addition to $\mathbf{k}_n^{q+1}$) the invariant cohomology class $\mathbf{k}_q^{r+1} \in H^{r+1}(\pi_q, q; \pi_r)$ and the Postnikov invariant $\mathbf{k}_{n \cdot q}^{r+1} \in H^{r+1}(T; \pi_r)$. Moreover, there is a natural embedding $i_q: K(\pi_q, q) \subseteq T$ and $\mathbf{k}_q^{r+1} = i_q^* \mathbf{k}_{n \cdot q}^{r+1}$. The author obtains a further invariant $\{\mathbf{k}_{n \cdot q}^{r+1}\}$ which is a coset of $H^{r+1}(T; \pi_r)$.

Let K be a complex, L a subcomplex of K, and f: $K^n \cup L \to Y$ a map admitting an extension to $K^{q+1} \cup L$. There is then a third obstruction to extending f to $K^{r+1} \cup L$ which is a coset of $H^{r+1}(K, L; \pi_r)$. This obstruction was described by N. Shimada and H. Uehara in a special case [Nagoya Math. J. **3** (1951), 67–72; MR **13**, 485]. In this paper the author expresses the obstruction in the general case in terms of the invariants $\mathbf{k}_n^{q+1}$, $\mathbf{k}_q^{r+1}$, $\{\mathbf{k}_{n \cdot q}^{r+1}\}$. The procedure is thus based in principle on that described by Eilenberg and MacLane [Ann. of Math (2) **60** (1954), 513–557; MR **16**, 392]. In particular, the relationship between cohomology classes of T and cohomology operations is elucidated. Thus, for $\mathbf{y} \in H^r(T; G)$ there is a γ-operation

$$\mathbf{y}_\gamma: H^{n_1}(K, L_1; \pi) \times H^{q_1}(K, L_2; \pi') \to H^{r-s}(K, L_1 \cup L_2; G)$$
$$(1 < n_1 \leqq n,\ 1 < q_1 \leqq q,\ s = n - n_1 + q - q_1)$$

(cf. $\mathbf{y}\vdash$ of Eilenberg-MacLane in the paper mentioned above) and a τ-operation $\mathbf{y}_\tau$, defined on the kernel of

$$\mathbf{k}_n^{q+1}\vdash: H^n(K; \pi) \to H^{q+1}(K; \pi')$$

such that $\mathbf{y}_\tau(\mathbf{x}_n)$ belongs to the factor group

$$H^r(K; G)/\{\mathbf{y}_\gamma(\mathbf{x}_n, H^q(K; \pi')) + i_q^* \mathbf{y}\vdash H^q(K; \pi')\}.$$

Then $\{\mathbf{k}_{n \cdot q}^{r+1}\}$ is the coset containing $\mathbf{k}_{n \cdot q}^{r+1}$ when $K = T$, $\mathbf{y} = \mathbf{k}_{n \cdot q}^{r+1}$, $G = \pi_r$, and $\mathbf{x}_n = \mathbf{b}_n$, the basic cohomology class in $H^n(T; \pi_n)$; briefly, $\{\mathbf{k}_{n \cdot q}^{r+1}\} = (\mathbf{k}_{n \cdot q}^{r+1})_\tau(\mathbf{b}_n)$. As usual, a homotopy classification theorem is deduced from the extension by considering $I \times K$.

Notations adopted in this paper are in several cases only explained in the earlier paper referred to. A somewhat perfunctory description of the cells of the complex T differs noticeably from that given in the earlier paper; but the difference is entirely due to misprints in the present paper.

P. Hilton (Manchester).

Fort, M. K., Jr. Essential mappings. Amer. Math. Monthly **63** (1956), 238–241.

Let $a_0, a_1, \cdots, a_n$ be distinct points in the plane. An example of a curve which is essential in the complement of $A = \{a_1, \cdots, a_n\}$ but inessential in the complement of any proper subset of A is presented and can be described as follows: Let U_k, $1 \leqq k \leqq n$, be a Jordan curve beginning at a_0 and containing a_i but no $a_j \neq a_i$ in its interior. Define f_k inductively by $f_1 = U_1$ and $f_{i+1} = f_i U_i f_i^{-1} U_i^{-1}$. Then f_n has the desired property. (It may be pointed out that this construction, carried out on the free group with generators $\{U_k\}$, provides an element $(\neq 1)$, of minimal length, which becomes 1 when any of the U_k is set equal to 1.) This paper, beginning with a discussion of strings and pegs, includes a readable introduction to homotopy and isotopy.

S. Stein.

★ **Segre, Beniamino. Recouvrements de sphères et correspondances entre variétés topologiques.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 149–175. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

Let F be a closed set on an n-sphere Σ_n and let μF be

the largest angle $\leqq\pi$ subtended at the center by two points of F. If R is a finite covering of Σ_n by closed sets F_i, let $\mu R = \max\{\mu F_i\}$. Let $\theta(n, v)$ be the lower bound of μR as R ranges over coverings of v closed sets each. In the first part of the paper, $\theta(n, v)$ is evaluated for certain values of n and v. It is shown that θ tends toward π as n and v tend toward ∞ with $v=O(n^k)$, $k>1$. In the second part, the author discusses the homology theory of correspondences between two n-dimensional varieties V and V' (taken in a definite order), a correspondence being defined by an n-cycle on $V\times V'$. A correspondence defines mappings of the homology groups of V into those of V' and therefore has a Lefschetz number. The correspondences between V and V' form a module. To each correspondence C between V and V' there corresponds an inverse C^{-1} which is a correspondence between V' and V. Correspondences belonging to a certain class called semi-regular can be composed with each other. Suppose C and C^{-1} are semi-regular. The correspondence CC^{-1} contains, in a sense, the identity correspondence I' of V' with V', taken a definite number α_1' of times. The Lefschetz number of $CC^{-1}-\alpha_1'\mathcal{I}'$ can be said to represent the algebraic sum of the ramification points of C. Call this number δ and let δ' be the corresponding number for C^{-1}. It is shown that $\delta-\delta'=\alpha_1\chi-\alpha_1'\chi'$, where χ, χ' are the Euler-Poincaré indices of V, V'. This formula generalizes a classical formula of Zeuthen for algebraic curves. *P. A. Smith* (New York, N.Y.).

See also: Zeeman, p. 939.

GEOMETRY

★**Kerékjártó, Béla. Les fondements de la géométrie. Tome I. La construction élémentaire de la géométrie euclidienne.** Akadémiai Kiadó, Budapest, 1955. 340 pp. 60.00 ft.

From the very modest introduction one would never guess that this is one of the richest works on the foundations of geometry. The introduction states that the author essentially follows Hilbert; it lists as major deviations only that angle does not appear as primitive concept, but is defined, and that the existence of motions (based on congruence of segments) is postulated instead of the first congruence theorem for triangles.

Actually, the book contains a wealth of unusual material. On the basis of the axioms of order and incidence we find Jordan's curve theorem for polygons in the plane and on convex polyhedra, Jordan's separation theorem for polyhedra in space (following Veblen), Euler's formula for convex polyhedra, an enumeration of all convex polyhedra which are regular from the point of view of incidence.

The next chapter (III) deals with congruence. It contains a complete discussion of motion (and orientation) of the plane and of space (from the point of view of absolute geometry). We find, in particular, the far too little known work of Hjelmslev on pencils of lines in the plane and bundles in space (Hjelmslev used these results for the introduction of ideal elements, which is not done here; therefore the words pencil and bundle are avoided and replaced by family). The chapter concludes with an enumeration of all metrically regular polyhedra and the finite groups of motions of space. Then (chapter IV) the Parallel Axiom is introduced. Besides subjects which one would expect, such as the euclidean theory of circles, similarity, introduction of coordinates etc., we find Hessenberg's derivation of Desargues' Theorem from Pappus' Theorem. The last chapter deals with continuity and does not contain any unusual results, although it is also more detailed than other works.

The exposition is meticulous and quite easily readable, except at a few places (like Jordan's Separation Theorem in space) where the subject matter is by its very nature involved. *H. Busemann* (Los Angeles, Calif.).

★**Hilbert, David. Fundamentos de la geometria. [The foundations of geometry.]** Instituto "Jorge Juan" de Matematicas, Madrid, 1953. vii+319 pp.

Translation of the 7th edition of Hilbert's Grundlagen der Geometrie [Teubner, Leipzig-Berlin, 1930].

★**Yaglom, I. M. Geometričeskie preobrazovaniya. I. Dviženiya i preobrazovaniya podobiya. [Geometrical transformations. I. Motions and similarity transformations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955 282 pp. 5.45 rubles.

This book, the first of two volumes, deals with motion (Chapter I) and similarities (Chapter II) of the euclidean plane. Each chapter begins with a discussion on what is geometry? The exposition is clear and simple, notable is the avoidance of technical terms, like collineation, in favor of terms more familiar or suggestive to the student; for example, "composition" (of transformations) is used instead of "product".

Half of the book consists of the text and the formulation of 106 problems, the second half gives solutions of the problems. These are partly novel and partly well-known theorems (e.g. the properties of the Euler line of a triangle) in problem form. *H. Busemann.*

Thébault, Victor. Sur un hexagone associé à un triangle. Mathesis **65** (1956), 70–77.

Cavallaro, Vincenzo G. Alcune brevi note sulla geometria del triangolo. Relazioni tra assi di coniche corrispondenti. Giorn. Mat. Battaglini (5) **3(83)** (1955), 211–222 (1956).

Lemmlein, V. G. On a geometrical problem for an n-dimensional simplex. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 215–218. (Russian)

In the n-dimensional Euclidean space E^n, let T_n be an n-dimensional simplex with vertices $M_1, M_2, \cdots, M_{n+1}$. Introduce the infinite periodic sequence $\{M_j\}$ by $M_{k+r(n+1)}=M_k$ $(r=1, 2, 3, \cdots; k=1, 2, \cdots, n+1)$. Let α be a fixed real number such that $|1+\alpha|>1$. Starting with an arbitrary point N_0 in E^n, define the sequence $\{N_j\}$ of points by $N_{j+1}=(N_j+\alpha M_{j+1})/(1+\alpha)$. It is shown that, for each $k=1, 2, \cdots, n+1$, $\lim_{r\to\infty} N_{k+r(n+1)}=P_k$ exists and is independent of the choice of N_0. If v_1 denotes the volume of T_n and v_2 denotes the volume of the n-dimensional simplex with vertices $P_1, P_2, \cdots, P_{n+1}$, then $v_2/v_1=\alpha^{n+1}/[(1+\alpha)^{n+1}-1]$. *Ky Fan.*

Bagemihl, F. A conjecture concerning neighboring tetrahedra. Amer. Math. Monthly **63** (1956), 328–329.

Durieu, M. Sur la parabole. Mathesis **65** (1956), 77–83.

Singh, Kuldip. Applications of the vertex of a great circle. J. Univ. Bombay. Sect. A. (N.S.) **24** (1955), no. 38, 19–22.

Clawson, J. W. A chain of circles associated with the n-line. Amer. Math. Monthly **63** (1956), 306–315.

Court, Nathan Altshiller. Sur quatre sphères réelles deux à deux orthogonales. Mathesis **65** (1956), 53–67.

Kostić-Lucin, V. G. The Cartesian ovals on the sphere. Univ. Beograd. Zb. Elektrotehn. Fak. **1954-1955**, 87–114. (Serbo-Croatian. French and English summaries)

Orts, J. Mª. Areas of certain enclosures. Gac. Mat., Madrid (1) **7** (1955), 125–129. (Spanish)

Schuster, Jan. Sur l'aire de l'ellipse. Mathesis **65** (1956), 40–46.

★ **Metody načertatel'noĭ geometrii i eë priloženiya. [Methods of descriptive geometry and their applications.] N. F. Četveruhin, editor.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 411 pp. 17.35 rubles.

This is a collection of 22 shorter and longer papers by different authors on a large variety of subjects in descriptive geometry. They are arranged in five chapters: 1) Geometrical transformations in descriptive geometry; 2) Descriptive geometry of more than three dimensions; 3) Special methods in descriptive geometry; 4) Descriptive geometry in Lobačevskiĭ space; 5) Mechanical constructions in descriptive geometry. There is thus a large variety of subjects, with a considerable amount of material presented, it seems, for the first time, such as the 57-page article by Z. A. Skopec on the principles of descriptive geometry in Lobačevskiĭ space, or the studies by Z. I. Pryanišnikova and T. B. Krunčak on the methods of E. S. Fedorov developed between 1907 and 1918. The references to the literature are also elucidating.

D. J. Struik (Cambridge, Mass.).

Železina, I. I. Line geometry of degenerate non-euclidean spaces. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 959–962. (Russian)

These degenerated non-euclidean spaces have a cone

$$\varepsilon_0(x^0)^2+\varepsilon_1(x^1)^2+\cdots+\varepsilon_m(x^m)^2=0$$

and a family of hyperplanes

$$\varepsilon_{m+1}u_{m+1}{}^2+\varepsilon_{m+2}u_{m+2}{}^2+\cdots+\varepsilon_n u_n{}^2=0$$

$$(1\leqq m\leqq n-2,\ \varepsilon=\pm 1).$$

For $n=4$ there are four cases: Π_3 with $(x_0)^2+(x^1)^2=0$, $u_2{}^2+u_3{}^2=0$; ${}^1\Pi_3$ with $(x^0)^2+(x^1)^2=0$, $u_2{}^2-u_3{}^2$; ${}^2\Pi_3$ with $(x^0)^2-(x^1)^2=0$, $u_2{}^2+u_3{}^2=0$ and ${}^3\Pi_3$ with $(x^0)^2-(x^1)^2$, $u_2{}^2-u_3{}^2=0$. In the case Π_3 the absolute consists of a pair of imaginary planes and a pair of imaginary points, etc. It is shown that the four-dimensional manifold of lines in these spaces (not counting those lines which intersect the absolute line) can be brought into one-to-one correspondence with the planes $R_2(\varepsilon)$ in case Π_3, $R_2(i)$ in cases ${}^1\Pi_3$ and ${}^2\Pi_3$, ${}^2R_2(\varepsilon)$ in case ${}^3\Pi_3$. There $R_2(\varepsilon)$ is the plane of the complex numbers $\alpha=(a+b\varepsilon)+i(c+d\varepsilon)$, $R_2(i)$ of $\alpha=(a+bj)+i(c+dj)$, ${}^1R_2(\varepsilon)$ of $\alpha=(a+bf)+\varepsilon(c+df)$; $i^2=j^2=-1$, $ij=ji$, $\varepsilon^2=f^2=+1$, $i\varepsilon=\varepsilon i$, $ef=fe$. Some observations on motions in these spaces (collineations which leave the absolute invariant) follow. The method outlined in this paper is familiar to those who have followed the ideas of B. A. Rozenfel'd, elaborated in "Non-Euclidean geometries" [Gostehizdat, Moscow, 1955, pp. 434ff.; MR **17**, 293].

D. J. Struik (Cambridge, Mass.).

Rios de Souza, Jayme Eduardo. Pohlke's theorem and its generalization to the case of central projection. Las Ciencias **18** (1953), 735–752. (Portuguese)

Kolmogorov, N. A. Analogues of double or anharmonic ratios for a space of three or more dimensions and their application to the proof of some theorems. Kirov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 7, 3–14. (Russian)

The biratio (i.e., the anharmonic ratio) of six non-coplanar points A, B, C, D, E, F, in ordinary space is defined to be the expression:

$$\frac{\text{vol. }\overline{ABEC}}{\text{vol. }\overline{ABED}}:\frac{\text{vol. }\overline{ABFC}}{\text{vol. }\overline{ABFD}},$$

the four terms of which are oriented volumes of tetrahedrons. The biratio is denoted by the symbol $(ABCDEF)$. In order to study this new symbol the author proves the relation:

$$\overline{ABDE}\cdot\overline{ABCF}+\overline{ABCD}\cdot\overline{ABEF}+\overline{ABCE}\cdot\overline{ABFD}=0,$$

thus generalizing a formula due to Monge [J. Ecole Polytech. **8** (1809), 68–117; M. Chasles, Traité de géométrie supérieure, Bachelier, Paris, 1852, art. 30, pp. 23–24; J. Neuberg, Mathesis (3) **7** (1907), 73–75, Lemma, p. 73]. With the help of this relation it is shown that if in $(ABCDEF)$ the letters A, B remain fixed, the 4! permutations of the symbol give rise only to six distinct values which are related to each other in the same way as values of the biratios of four collinear points.

The final result arrived at is that the 6! permutations of the new symbol give rise to $6!/4\cdot 2!=90$ distinct values, and of those only three are unrelated, the remaining values being functions of those three, and may be readily computed.

If in $(ABCDEX)=a_1$ the first five points are fixed and a_1 is a given constant, the variable point X lies in a fixed plane, the locus of points the ratio of whose distances from the two fixed planes ABC, ABD is constant. The above equality may thus be considered as the equation of the plane ABX.

The three simultaneous equations:

$$(ABCDEX)=a_1,\ (BCDEXA)=b_1,\ (CDEXAB)=c_1,$$

where the points A, B, C, D, E, are fixed, and a_1, b_1, c_1, are three given unrelated constants, determine uniquely the position of X as the point of intersection of the three planes ABX, BCX, CDX. The constants a_1, b_1, c_1, may be said to be the projective coordinates of X with reference to the five fixed points.

If M_i $(i=1, 2, \cdots, n, n+1, n+2, n+3)$ are $n+3$ points in an n-dimensional space, their biratio in that space is, by definition, the expression:

$$\frac{\text{hypervol. }\overline{(M_1M_2\cdots M_nM_{n+2})}}{\text{hypervol. }\overline{(M_1M_2\cdots M_nM_{n+3})}}:\frac{\text{hypervol. }\overline{(M_1M_2\cdots M_{n+1}M_{n+2})}}{\text{hypervol. }\overline{(M_1M_2\cdots M_{n+1}M_{n+3})}}$$

and denoted symbolically by

$$(M_1M_2\cdots M_nM_{n+1}M_{n+2}M_{n+3}),$$

the terms of the fractions being oriented hypervolumes of

n-dimensional simplexes. [Note. If the definition of $(ABCDEF)$ given above is to conform to the present generalization, the positions of the two pairs of points C, D and E, F should be interchanged; this change, however, does not affect the value of the symbol.]

The author states that it may be proved without much trouble that of the $(n+3)!$ possible permutations of the symbol only $(n+3)!/4(n-1)!$ have different values, and of those only n are independent of one another, and the rest are functions of the independent values.

If we consider the n simultaneous equations:

$$(M_1M_2\cdots M_nM_{n+1}M_{n+2}X)=m_1,$$
$$(M_2M_3\cdots M_nM_{n+1}M_{n+2}XM_1)=m_2,$$
$$\cdots$$
$$(M_nM_{n+1}M_{n+2}XM_1M_2\cdots M_{n-1})=m_n,$$

where X is a variable point, the other $n+2$ points being fixed, and $m_1, m_2, \cdots, m_n$ are n given unrelated constants, those equations determine the position of X uniquely as the point of intersection of n hyperplanes. Hence the m's may be considered as the coordinates of X in the n-space.

The author uses the new symbol to obtain analogs of Ceva's and Menelaus' theorems for a four-dimensional simplex, and to determine the intersection of that simplex with a hypersphere. *N. A. Court* (Norman, Okla.).

Kolmogorov, N. A. Properties of systems of points lying on a straight line, on a circle and on an equilateral hyperbola. Kirov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 7, 15–28. (Russian)

M. Chasles established an identity involving two sets of collinear points A_i $(i=1, 2, \cdots, n)$ and M_i $(i=1, 2, \cdots, n-1)$, and used this formula to obtain a decomposition of rational algebraic fractions into simple fractions [Traité de géométrie supérieure, Bachelier, Paris, 1852, Chap. XVI, arts. 318, 326]. The author considers the identity (also given by Chasles, ibid., art. 319)

$$\frac{A_1M_1\cdot A_1M_2\cdots A_1M_{n-2}}{A_1A_2\cdot A_1A_3\cdots A_1A_n}+\frac{A_2M_1\cdot A_2M_2\cdots A_2M_{n-2}}{A_2A_3\cdot A_2A_4\cdots A_2A_1}$$
$$+\frac{A_3M_1\cdot A_3M_2\cdots A_3M_{n-2}}{A_3A_4\cdot A_3A_5\cdots A_3A_1\cdot A_3A_2}+\cdots=0$$

involving n points A_i and $n-2$ points M_i. This identity the author turns into the following sum of products of biratios (i.e. anharmonic ratios):

$$(A_1A_nM_1A_2)(A_1A_nM_2M_3)(A_1A_nM_3A_4)\times\cdots$$
$$\times(A_1A_nM_{n-2}A_{n-1})+(A_2A_nM_1A_1)(A_2A_nM_2A_3)\times$$
$$(A_2A_nM_3A_4)\cdots(A_2A_nM_{n-2}A_{n-1})+\cdots+(A_{n-1}A_nM_1A_1)$$
$$\times(A_{n-1}A_nM_2A_2)\cdots(A_{n-1}A_nM_{n-2}A_{n-2})=1.$$

If all the points A_i and M_i are projected from a point 0, each of those biratios may be stated in terms of the sines of the angles formed by the corresponding projecting lines; those angles, in turn, may be replaced by the arcs which their sides determine on a circle (O) having O for center. Now if a fixed origin is taken on (O) and the abscissas of the traces of the lines OA_i, OM_i on (O) are denoted by a_i, m_i, the latter identity, after some small modifications, may be stated as follows:

$$\frac{\sin(m_1-a_1)\sin(m_2-a_1)\sin(m_3-a_1)\cdots}{\sin(a_2-a_1)\sin(a_3-a_1)\sin(a_4-a_1)\cdots}$$
$$+\frac{\sin(m_1-a_2)\sin(m_2-a_2)\sin(m_3-a_2)\cdots}{\sin(a_3-a_2)\sin(a_4-a_2)\sin(a_5-a_2)\cdots}+\cdots=0.$$

We may think of A_n as being a variable point, while the remaining points remain fixed. If in the above identity we replace a_n by x, the resulting expression may be cast in the following final form:

$$\frac{\prod_1^n\sin(x-m_i)}{\prod_1^{n+1}\sin(x-a_i)}=$$
$$\frac{A_1}{\sin(x-a_1)}+\frac{A_2}{\sin(x-a_2)}+\cdots+\frac{A_n}{\sin(x-a_n)},$$

where

$$A_1=\frac{\prod_1^n\sin(a_1-m_i)}{\prod_1^{n+1}\sin(a_1-a_i)},\text{ etc.}$$

The analogy of this result with the one obtained by Chasles is obvious, and so are its merits from the point of view of the integral calculus. Moreover, the author points out that an analogous formula involving hyperbolic sines is equally valid, since he proved in the introductory part of the paper (p. 18, Th. 3) that if instead of the circle (O) we had drawn an equilateral hyperbola (H) of center O, it would be legitimate, in connection with the biratios considered, to use the hyperbolic sines of the arcs involved.

In the introductory part of the paper there is given the formula:

$$\sin h(\overline{AD}/2)\cdot\sin h(\overline{BC}/2)+\sin h(\overline{AB}/2)\cdot\sin h(\overline{CD}/2)+$$
$$\sin h(\overline{AC}/2)\cdot\sin h(\overline{DB}/2)=0$$

for a quadrilateral $ABCD$ inscribed in a equilateral hyperbola (analog of Ptolemy's theorem), and several other formulas of this kind. The paper concludes with the derivation of Mercator's identity from Chasles' identity mentioned before. *N. A. Court* (Norman, Okla.).

Brauner, H. Über die ähnlichen und sich ähnlich projizierenden Kegelschnitte auf Quadriken. Arch. Math. **7** (1956), 78–86.

Given two conics

$$k(x)\equiv\sum_1^3 a_{ik}x_ix_k=0\text{ and }l(x)\equiv\sum_1^3 b_{ik}x_ix_k=0$$

in the projective plane. Let $K(u)\equiv\sum_1^3 A_{ik}u_iu_k=0$ and $L(u)\equiv\sum_1^3 B_{ik}u_iu_k=0$ be their equations in line coordinates. The conic $k(x)+\lambda l(x)$ then has an equation in line coordinates $K(u)+2\lambda H(u)+\lambda^2L(u)=0$. The cross-ratio δ of the four points in which the line $\sum_1^3 u_ix_i=0$ intersects k and l satisfies the equation

$$(1)\qquad(\delta-1)^2H^2(u)-(\delta+1)^2K(u)L(u)=0.$$

Thus the straight lines whose intersections with k and l have the given cross-ratio δ envelop the curve (1). In general it is of class 4 and order 12.

Let Φ be a quadric. Let k be the intersection of Φ with the infinite plane π_∞ and let l be the absolute conic [let π and $\bar\pi$ be two parallel planes and let O be a point in $\bar\pi$. Let k be the intersection of Φ with $\bar\pi$ while l consists of the isotropic lines in $\bar\pi$ through O]. The ∞^2 planes whose intersections with Φ are [projected from O onto conics in π] similar to a given conic intersect k and l with a fixed cross-ratio. Thus the discussion of (1) can be applied to the curve in π_∞ [in $\bar\pi$] which these planes envelop [in general the curve in $\bar\pi$ is of order 8]. *P. Scherk.*

Bennhold, Friedrich. Zur synthetischen Begründung der projektiven Geometrie der Ebene. Math. Ann. **129** (1955), 213–229.

The work of R. Moufang has been responsible for many

investigations in recent years. Assuming the uniqueness of the fourth harmonic point she was able to deduce the "Little Desargues Theorem" which is equivalent to the restricted form of Pappus' Theorem. If one assumes the full Desargues Theorem and an axiom of separation which establishes an order on the projective line along with an Archimedian postulate, one can deduce the full Pappus Theorem and continuity. The present author modifies this procedure to suit the Moufang case. The method utilizes to the full the properties of Möbius nets, but avoids the customary algebraic interpretations.

G. de B. Robinson (Toronto, Ont.).

Alekseeva, G. P. Topological classification of collineations of a projective plane. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 37–41. (Russian)

Two collineations A, B of P^2 are topologically equivalent if a topological mapping Γ of P^2 on itself exists such that $B=\Gamma A\Gamma^{-1}$. A complete set of topologically non-equivalent collineation of P^2 is given.

H. Busemann (Los Angeles, Calif.).

Mokriščev, K. K. On solvability of construction problems of the second degree in the Lobačevskiĭ plane by means of the horocycle. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 515–519. (Ukrainian. Russian summary)

It is shown that every second-order construction problem in the hyperbolic plane can be solved by means of limit circles alone in this sense: if a point of the limit circle and two points of a radius are known, then it is assumed that the limit circle can be drawn.

H. Busemann (Los Angeles, Calif.).

Kiriščev, R. I. On a theorem of D. D. Morduhaĭ-Boltovskoĭ. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 207–208. (Russian)

In negation of the "theorem" to which the title refers [In memoriam N. I. Lobatschevskiĭ, v. 2, Glavnauka, Kazan, 1927, pp. 67–82] it is shown that every second-order construction problem in the hyperbolic plane can be solved by means of a ruler and a compass of fixed opening. Instead of the compass a limit circle or a fixed equidistant curve can be used. *H. Busemann.*

Švarc, A. S. Topological and infinitesimal classification of affine transformations. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 9–26. (Russian)

Two mappings A, B of the space R are topologically equivalent (equimorph or infinitesimally equivalent) if a topological (1-1 both ways uniformly continuous) mapping Γ of R on itself exists such that $B=\Gamma A\Gamma^{-1}$. First the affinities of E^n with diagonal matrices and all affinities of E^3 are topologically classified. Then necessary conditions for infinitesimal equivalence of affinities of E^n are derived. Finally all infinitesimally non-equivalent classes of affinities of E^2 and E^3 are enumerated. Whereas the result for E^3 is too involved for a review, the result for E^2 is simple: Two affinities of A, B of E^2 are equimorph if and only if either A and B are affinely equivalent, or A and B are affinely equivalent to two rotations followed by dilations with the same factor from the centers of the rotation. *H. Busemann* (Los Angeles, Calif.).

Hazay, I. Untersuchungen über die Projektion zwischen Ellipsoiden und über die Bestimmung der relativen Lage der Ellipsoide durch Projektion. Acta Tech. Acad. Sci. Hungar. **14** (1956), 165–194. (Russian, English and French summaries)

Hazay, I. Beiträge zur Bestimmung der Konstanten bei der Projektion zwischen Ellipsoiden und der Lage der Ellipsoide. Acta Tech. Acad. Sci. Hungar. **14** (1956), 401–423. (Russian, English and French summaries)

Rinner, K. Eine allgemeine analytische Lösung des Folgebildanschlusses. Österreich. Z. Vermessgswes. **44** (1956), 4–9.

See also: Aleksandrov, p. 931; Bruck and Paige, p. 943; Stoka, p. 955.

Convex Domains, Integral Geometry, Metric Geometry

Hadwiger, H. Volumschätzung für die einen Eikörper überdeckenden und unterdeckenden Parallelotope. Elem. Math. **10** (1955), 122–124.

A right parallelotope is the set of all those points in euclidean k-space whose coordinates z_i in a suitable rectangular cartesian coordinate system satisfy $a_i\leq z_i\leq b_i$, $i=1, \cdots, k$. Let $V(A)$ denote the volume of the closed bounded convex set A. The author proves by induction: To every A there are two right parallelotopes P and Q such that $Q\subset A\subset P$ and $k^kV(Q)\geq V(A)\geq(k!)^{-1}V(P)$. [Reviewer's remark: The author's construction of Q shows that there is a 2^k-hedron $R\subset A$ whose diagonals are mutually perpendicular such that $k!V(R)\geq V(A)$.]

P. Scherk (Kingston, Ont.).

★ **Montel, Paul. Sur la géométrie finie et les travaux de M. C. Juel.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 9–26. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

This is a reprint of an article published in Bull. Sci. Math. (2) **48** (1924), 109–128. *P. Scherk.*

Aleksandrov, A. D.; and Sen'kin, E. P. Supplement to the paper, "On the rigidity of convex surfaces". Vestnik Leningrad. Univ. **11** (1956), no. 1, 104–106. (Russian)

This note was intended to be part of the article mentioned in the title [same Vestnik **10** (1955), no. 8, 3–13; MR **17**, 295]. It establishes the corresponding result for polyhedra: Let P_1, P_2 be two intrinsically isometric convex polyhedra with boundaries. Assume there are points z_1, z_2 such that 1) the distances of z_1, z_2 to points on the boundaries of P_1, P_2 corresponding under the isometry are equal, 2) P_1 and P_2 are seen from z_1 and z_2 either from the inside or both from the outside, and 3) there is a plane through z_i such that P_i lies entirely on one side of this plane. Under these conditions P_1 and P_2 are congruent. *H. Busemann* (Los Angeles, Calif.).

Leja, F. Sur la distribution des point extrémaux dans les ensembles plans. Ann. Mat. Pura Appl. (4) **39** (1955), 143–146.

If the bounded frontier F of an unbounded domain D be divided into two disjoint closed parts s and $F-s$, then the proportion α_n of a set of n extremal points for F which lie on s tends to a limit as $n\to\infty$. The n points of a set are called extremal if the product of their mutual distances is maximum. *H. D. Ursell.*

Pi Calleja, Pedro. On the polar of a given figure with respect to a circle with center at the barycenter. Math. Notae **9** (1949), 88–93 (1950). (Spanish)

Let the origin of a system r, φ of polar coordinates lie inside the plane convex curve C and denote the supporting function of C by $h(\varphi)$. In answer to a question of Santaló it is shown that the relations

$$\int_0^{2\pi} \cos\varphi h^{-3}(\varphi)\ d\varphi = \int_0^{2\pi} \sin\varphi h^{-3}(\varphi)\ d\varphi = 0$$

do not imply that the origin is the center of gravity of the domain bounded by C, for example, when C is a quadrangle. If C has a center or is a triangle, then the answer is affirmative. Relations between the curvatures of C and the polar reciprocal $r=h(\varphi)$ of C are also given, which are, however, known from Minkowskian geometry [see (13) in Busemann, Ann. Mat. Pura Appl. (4) **31** (1950), 281–295; MR **13**, 74]. *H. Busemann* (Los Angeles, Calif.).

Santaló, L. A. Integral geometry in three-dimensional spaces of constant curvature. Math. Notae **9** (1949), 1–28 (1950). (Spanish)

The cinematic density for three-dimensional hyperbolic and elliptic space is obtained, and from it densities for planes and straight lines are deduced. The main result is the cinematic "Hauptformel", which surprisingly enough, in contrast to the two-dimensional case, is independent of the curvature of the space and thus identical with the euclidean formula, i.e.: Let Q_i ($i=0, 1$) be a bounded set of volume V_i whose boundary has everywhere a continuous Gauss curvature and has area F_i. If N_0', N_1', $N'(Q_0, gQ_1)$ are the Euler characteristics of Q_0, Q_1, and of the intersection of Q_0 with the image gQ_1 of Q_1 under the motion g, then

$$\int N'(Q_0, gQ_1)dg = 8\pi^2(N_1'V_0+N_0'V_1)-2\pi(M_0F_1+M_1F_0),$$

where M_i is the integral of the mean curvature of the boundary of Q_i. Steiner's formula for parallel surfaces follows (already given for n dimensions by Allendoerfer [Bull. Amer. Math. Soc. **54** (1948), 128–135; MR **9**, 607]. Finally, results of Varga on the intersection of moving surfaces in E^3 are extended [Math. Z. **41** (1936), 768–784]. The only change is the replacement of the absolute curvatures by the curvatures relative to the ambient space. *H. Busemann* (Los Angeles, Calif.).

Busemann, Herbert; and Pedersen, Flemming P. Tori with one-parameter groups of motions. Math. Scand. **3** (1955), 209–220 (1956).

Let R be a two-dimensional torus, metrized as a G-space [Busemann, The geometry of geodesics, Academic Press, New York, 1955, MR **17**, 779], which possesses a one-parameter group of isometries Γ_R. If no orbit of Γ_R is a closed geodesic, then the metric is Minkowskian (flat, if R is Riemannian), and there is a transitive abelian group of motions. In the universal covering plane P (with lifted metric) call a geodesic straight if it is isometric in the large to a euclidean straight line. If every orbit of Γ_R is a geodesic, all geodesics of P are straight. (Theorem 4.5 is independent of and properly precedes 4.4.) Finally, if Γ_R possesses some (but not all) closed geodesic orbits, P consists of strips bounded by certain straight lines such that geodesics crossing or asymptotic to these boundaries are the only straight lines of P. This description of the conjugate points contains most of the qualitative part of Bliss [Ann. of Math. (2) **4** (1902), 1–21], Kimball [Amer. J. Math. **52** (1930), 29–52], and Hedlund [Ann. of Math. **33** (1932), 719–739]. *L. W. Green* (Minneapolis, Minn.).

Perkal, J. Sur les ensembles ε-convexes. Colloq. Math. **4** (1956), 1–10.

For a positive number ε and a subset E of a metric space M, the ε-convex hull $C_\varepsilon(E)$ of E is defined as $M\sim C_\varepsilon'(E)$, where $C_\varepsilon'(E)$ is the union of all open spheres of radius $\frac{1}{2}\varepsilon$ in $M\sim E$. Equivalently, $p \in C_\varepsilon(E)$ if and only if $|p, q|\geqq\frac{1}{2}\varepsilon$ whenever $q \in M$ and $|q, E|\geqq\frac{1}{2}\varepsilon$. The set E is called ε-convex provided $E=C_\varepsilon(E)$. The paper establishes various properties of the operations C_ε, of which the following are typical: $C_\varepsilon[C_\varepsilon(E)]=C_\varepsilon(E)=C_\varepsilon(\bar{E})$; the intersection of a family of ε-convex sets is ε-convex; $\lim_{\varepsilon\to 0} C_\varepsilon(E)$ is the closure of E; if M is a Euclidean space and E a domain in M, then $\lim_{\varepsilon\to\infty} C_\varepsilon(E)$ is the closed convex hull of E; $C_\varepsilon(E)$ is ε-connected if and only if E is ε-connected. There is stated the following problem of Borsuk: In a Euclidean space, must every ε-convex set be a polytope, or at least locally contractible? *V. L. Klee.*

See also: Sudan, p. 948; Baiada, p. 963.

Differential Geometry

Ostrowski, Alexander. Mathematische Miszellen. XXIII. Zur Differentialgeometrie der ebenen Parallelkurven. Jber. Deutsch. Math. Verein. **58** (1955), Abt. 1, 49–53.

Given a plane rectifiable curve S with the arc length s and the parametric representation $x=x(s)$, $y=y(s)$. Suppose $\dot{x}=dx/ds$ and $\dot{y}=dy/ds$ exist and the angle $t=t(s)$ between the directed tangents of S and the positive x-axis exists and is a strictly monotonic function of s. Then $\dot{x}$, $\dot{y}$, and t are readily seen to be continuous. We form a closed interval which contains the reciprocals of all the four derivates of $t(s)$ and choose any real number l outside of this interval. Then $\sigma^*=s-lt$ will be a strictly monotonic function of s. Put $\varepsilon=1$ [$=-1$] if σ^* is increasing [decreasing]. The parallel curve S_l: $\xi=x-l\dot{y}$, $\eta=y+l\dot{x}$ has the arc length $\varepsilon\sigma^*$ and satisfies $d\xi/d\sigma^*=\dot{x}$ and $d\eta/d\sigma^*=\dot{y}$. S_l has a directed tangent. It forms the angle t [$t+\pi$] with the positive x-axis if σ^* is increasing [decreasing]. — The methods are similar to those of an earlier paper [Arch. Math. **6** (1955), 170–179; MR **17**, 295]. *P. Scherk* (Kingston, Ont.).

Gołąb, S. On the concept of the centre of the second curvature and on a generalization of a certain geometrical meaning of v. Lilienthal. Ann. Polon. Math. **2** (1955), 215–218 (1956).

Let $x(\sigma)$ denote a regular curve in euclidean n-space with the arc length σ and the moving n-hedron $t_1, \cdots, t_n$. Thus its curvatures $k_0=0$, $k_1, \cdots, k_{n-1}$, $k_n=0$ satisfy Frenet's formulas $dt_i/d\sigma=-k_{i-1}t_{i-1}+k_it_{i+1}$ ($i=1, \cdots, n$). Let l_i denote the straight line through $x(0)$ parallel to $t_i(0)$. Let $1<j<n$. Construct the straight line through $x(0)-\sigma t_{j+1}(0)$ parallel to $t_j(\sigma)$ and project it into the plane through l_j and l_{j+1}. If $k_j(0)\neq 0$, the intersection of this projection with l_j will converge to $x(0)+k_j(0)^{-1}t_j(0)$ as σ tends to zero. *P. Scherk* (Kingston, Ont.).

Gołąb, S. On the geometrical significance of curvatures of higher orders for curves lying in n-dimensional spaces. Ann. Polon. Math. **2** (1955), 209–214 (1956).

[Cf. the preceding review.] Let δ_j [δ_{n-1}^*] denote the

[oriented] distance of $x(\sigma)$ from the subspace spanned by $l_1, \cdots, l_j$ [by $l_1, \cdots, l_{n-1}$]; $\delta_0=|x(\sigma)-x(0)|$. Then

$$\lim_{\sigma\to 0}\frac{\delta_j}{\delta_0\delta_{j-1}}=\frac{|k_j(0)|}{j+1}\ (j=1, \cdots, n-2), \tag{1}$$

$$\lim_{\sigma\to 0}\frac{\delta_{n-1}^*}{\delta_0\delta_{n-2}}=\frac{k_{n-1}(0)}{n}. \tag{2}$$

[Formula (1) can be improved to

$$\lim_{\sigma\to 0}\frac{\delta_j}{|\sigma|^{j+1}}=\frac{|k_1(0)\cdots k_j(0)|}{(j+1)!}\ (j=0, 1, \cdots, n-1).$$

The reviewer computed

$$\lim_{\sigma\to 0}\frac{\delta_{n-1}^*}{\sigma^n}=\frac{k_1(0)\cdots k_{n-1}(0)}{n!}.$$

This agrees with (2) only if $\sigma_1 k_1(0)\cdots k_{n-2}(0)\geqq 0$.]

P. Scherk (Kingston, Ont.).

Bilinski, Stanko. Eine Verallgemeinerung der Formeln von Frenet und eine Isomorphie gewisser Teile der Differentialgeometrie der Raumkurven. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 175–180. (Serbo-Croatian summary)

Let $\varkappa_1, \tau_1, \xi_1, \xi_2, \eta_1$ be the curvature, torsion, tangent vector, principal normal vector, and binormal vector of a curve in Euclidean 3-space, all referred to the arc length. The generalization referred to in the title is set up by the recursion formulas

$$\varkappa_{i+1}=\sqrt{(\varkappa_i^2+\tau_i^2)},\ \tau_{i+1}=\frac{\varkappa_i\dot{\tau}_i-\varkappa_i\tau_i}{\varkappa_i^2+\tau_i^2},\ \xi_{i+1}=\eta_i\times\xi_i,$$

$$\eta_{i+1}=\frac{\tau_i\xi_i+\varkappa_i\eta_i}{\sqrt{(\varkappa_i^2+\tau_i^2)}},\ i=1, 2, \cdots.$$

The vectors ξ_i, ξ_{i+1}, η_i constitute the ith moving trihedral; they satisfy equations analogous to the Frenet equations. From a knowledge of $\varkappa_i$ and τ_i for a fixed i as functions of s, together with suitable initial conditions, one gets $\varkappa_1$ and τ_1 and hence the curve itself. From any theorem of ordinary curve theory which depends entirely on the Frenet equations one can get new theorems by replacing $\varkappa_1, \tau_1, \xi_1, \xi_2, \eta_1$ by $\varkappa_i, \tau_i, \xi_i, \xi_{i+1}, \eta_i$. For instance, in this way one goes from the theorem that a curve whose tangent vectors make a fixed angle with a fixed direction has $\varkappa_1/\tau_1$=constant, and conversely (helix) to the theorem that a curve whose principal normals make a fixed angle with a fixed direction has $\varkappa_2/\tau_2$ constant, and conversely. One such curve has natural equations $\varkappa_1=c|\cos s|$, $\tau_1=c\sin s$. *A. Schwartz* (New York, N.Y.).

Şemin, F. On an extension of Liouville's formula. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 91–94. (Turkish summary)

The classical formula of Liouville is extended to the case of a curve C and a non-orthogonal system of coordinate curves on a surface. Let $g, g^*, \tilde{g}$ and ds_1, ds_2, ds denote geodesic curvatures and linear elements of the u-curve, v-curve, C, respectively, at a point P, and let ν, ν^* denote the angles between the tangent to C at P and the tangents at P to the u-curve, v-curve, respectively. The result is expressed by the relation

$$\tilde{g}=[(g+\nu_1)\sin\nu^*+(g^*-\nu_2^*)\sin\nu]/\sin(\nu+\nu^*)$$

in which $\nu_1=d\nu/ds_1$ and $\nu_2^*=d\nu^*/ds_2$. *P. O. Bell.*

Wunderlich, Walter. Doppelloxodromen mit schneidendem Achsenpaar. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. **164** (1955), 17–34.

A loxodrome is a curve which intersects the planes through a fixed axis at a constant angle. The author determines the real curves which are loxodromes in two ways and whose two axes are collinear. His discussion includes the special cases that these axes are conjugate complex or parallel or that one of them lies in the infinite plane. *P. Scherk* (Kingston, Ont.).

Hayashi, Yoshio. On the fundamental quantities of the holomorphic surface. Sci. Papers Fac. Engrg. Tokushima Univ. **3** (1951), no. 2 (new no. 4), 58–59.

Hayashi, Yoshio. On the geometrical properties of holomorphic function. Sci. Papers Fac. Engrg. Tokushima Univ. **1953**, no. 5, 61–63. (Japanese. English summary)

Gönenç, Süeda. Cinematica generalizzata. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 17–47. (Turkish summary)

The article deals with the kinematics of inversive geometry in the plane, as developed by Blaschke. The author makes use of complex numbers, studying

$$w=(Aw'+B)(Cw'+D)^{-1},$$

the coefficients being functions of t. The euclidean case $A=e^{it}$, $B=b_1+b_2i$, $C=0$, $D=1$ is generalized. The two instantaneous rotation centers are dealt with and also the case that these poles coincide. Other topics are: the osculating circle of the trajectory, acceleration of an arbitrary point, the use of tetracyclic coordinates and quaternions. *O. Bottema* (Delft).

Tihonov, V. A. On bending on a spherically conjugate basis. Trudy Voronež. Gos. Univ. Fiz.-Mat. Sb. **27** (1954), 79–83. (Russian)

A net on a surface which is harmonic with respect to some net of curves of constant normal curvature R^{-1} is called spherically conjugate. The lines of curvature are always spherically conjugate. The paper studies continuous bending (in the small) of a surface on a spherically conjugate basis, which means that a spherically conjugate net is preserved under the bending process. A non-trivial example is provided by the surfaces of constant mean curvature which can be bent in this way with the isotropic curves as spherically conjugate net. *H. Busemann.*

Saban, Giacomo. Su due invarianti integrali della teoria metrica delle superficie rigate. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 50–54.

L'auteur revient ici sur certains résultats établis par lui dans une note antérieure [mêmes Rend. (8) **5** (1948), 24–30; MR **10**, 399], et dans laquelle, grâce à l'emploi des vecteurs duaux, et se basant sur des résultats sur les courbes gauches fermées de B. Segre, il établit certaines limitations pour les invariants intégraux fondamentaux de la théorie des surfaces réglées. Ces résultats sont ici améliorés, et, à la faveur d'une relation entre les limites inférieures des invariants intégraux en question, l'auteur parvient aux deux inégalités

$$\int_L p\ ds\geqq 2\pi\cos r_3,\ \int_L |q|\ ds\geqq 2\pi r_1.$$

où L est la ligne de striction de la surface réglée (fermée) envisagée d'arc s, où $p=\sqrt{(dg/ds)^2}$, $|q|=\sqrt{(dt/ds)^2}$, $\mathbf{g}$ et $\mathbf{t}$ étant les vecteurs unitaires portés par la génératrice de la surface et par la tangente au point central normale à cette génératrice, r_1 étant le rayon sphérique de la plus petite calotte contenant l'image sphérique de la surface, et r_3 celui de la plus petite calotte contenant l'image sphérique de la surface réglée engendrée par les tangentes $\mathbf{t}$ définies plus haut. *P. Vincensini* (Marseille).

Gorowara, K. K. Skewness of distribution of the generators of certain ruled surfaces and their parameters of distribution. Rev. Fac. Sci. Univ. Istanbul. Sér. A. 18 (1953), 223–230. (Turkish summary)

A_1, A_2, A_3 étant respectivement, la génératrice d'une surface réglée S_1, la normale à S_1 au point central M de A_1, et la perpendiculaire en M à A_1 et A_2, l'auteur envisage une droite A_1^* issue de M dont les cosinus directeurs relatifs à (A_1, A_2, A_3) sont des fonctions de l'arc de la ligne de striction de S. Il étudie pour la surface réglée $S_1^*(A_1^*)$ et pour les surfaces associées A_2^*, A_3^* analogues de A_1, A_2, les paramètres de distribution et les quantités (skewness) $(\mathbf{a}, \mathbf{a}', \mathbf{a}'')/(\mathbf{a}'^2)^{3/2}$, où $\mathbf{a}$ est le vecteur unitiare porté par la génératrice rectiligne de la surface réglée envisagée, et $\mathbf{a}'$, $\mathbf{a}''$ ses derivés par rapport au paramètre fixant la génératrice. Faisant usage de la théorie des vecteurs duaux, et utilisant des formules de Blaschke sur ce sujet, il est conduit à un certain nombre de résultats relatifs aux surfaces engendrées par les droites A_1^* situées dans les plans centraux de S_1 et coupant A_1 sous un angle constant. *P. Vincensini* (Marseille).

Švec, Alois. Sur la déformation projective des surfaces réglées. Czechoslovak Math. J. **5(80)** (1955), 355–361. (Russian summary)

E. Čech a introduit la notion de transformations K-linéarisantes de correspondances entre deux espaces projectifs [Časopis Pest. Mat. Fys. **74** (1949), 32–48; MR 12, 534]. Dans le présent mémoire, l'auteur généralise ces transformations; il considère une correspondance quasiasymptotique T entre deux surfaces réglées Π, $\bar{\Pi}$ plongées dans un espace de dimension impaire $2n+1$, dans laquelle se correspondent les génératrices; il définit une droite K-linéarisante en tout point de chaque surface. La correspondance T est dite du type $T_{p,q}$ si en chaque point de Π la droite K-linéarisante appartient à l'union de l'espace p-tangent de la surface Π le long de la génératrice au point considéré et du $q$ème espace osculateur à la courbe quasiasymptotique en ce point. En particulier, une correspondance $T_{-1,0}$ définit une déformation projective d'ordre $n+1$ en ce sens que pour chaque couple de génératrices, il existe une collinéation K qui réalise un contact analytique d'ordre $n+1$ des deux surfaces tout le long de ces génératrices. *M. Decuyper* (Lille).

Čech, Eduard. Deformazioni di congruenze di rette. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 55–66.

A. Terracini a introduit la notion d'élément linéaire projectif d'une congruence L, et a montré que l'invariance de cet élément linéaire relativement à une transformation T ($L \to L'$) caractérise, à une dualité eventuelle près, les déformations projectives de la congruence envisagée. E. Čech décompose en quatre formes élémentaires l'élément linéaire projectif de Terracini, et l'invariance de ces formes par une transformation T (l'invariance de trois d'entre-elles entraînant celle de la quatrième) suffit pour que T soit une déformation projective. L'auteur analyse les rôles respectifs des formes introduites; il attache à l'invariance de chacune d'elles un type particulier de déformation [déformations ponctuelles, planaires, focales de 1ère ou de 2ème espèce, asymptotiques de 1ère ou de 2ème espèce], et donne les constructions géométriques des six types de déformations obtenus. Pour chacun de ces six types la congruence L peut être choisie arbitrairement, la transformé L' dépendant alors d'une fonction arbitraire de deux variables. Les cas de réalization simultanée de deux des six types précédents (déformations doubles) et les circonstances géométriques qui les accompanent sont spécialement envisagés, de même que les relations qui existent entre eux et la déformation projective. *P. Vincensini* (Marseille).

Vasil'ev, A. M. On a pair of W congruences. Moskov. Gos. Univ. Uč. Zap. **163**, Mat. 6 (1952), 137–145. (Russian)

This is an investigation of pairs of line congruences with an established correspondence of lines, satisfying the conditions: 1) both congruences are W-congruences; 2) the asymptotic lines correspond on all four focal surfaces of the pair, 3) the osculating quadrics to one family of ruled surfaces of one congruence, together with the corresponding asymptotic lines of the focal surfaces, are also the osculating quadrics to the family of such surfaces of the second congruence, which correspond to the second family of the first congruence. *D. J. Struik.*

Bol, Gerrit. Projektive und affine Eigenschaften des Darbouxschen Flächenkranzes. Abh. Math. Sem. Univ. Hamburg **20** (1955), 64–96.

Small German letters denote points in centro-affine three-space. Its origin O is fixed throughout. Capital German letters $\mathfrak{A}, \cdots$ denote planes $\mathfrak{A}\mathfrak{x}+1=0, \cdots$. A pair $\mathfrak{p}$, $\mathfrak{P}$ defines a vector $\mathfrak{p}=[\mathfrak{p}, \mathfrak{P}]$ in six-space. If $\mathfrak{q}=[\mathfrak{q}, \mathfrak{Q}]$, put $\mathfrak{p}\circ\mathfrak{q}=\mathfrak{P}\mathfrak{q}+\mathfrak{Q}\mathfrak{p}$. The straight line $(\mathfrak{v}, \mathfrak{w})$ through the points $\mathfrak{v}\neq\mathfrak{w}$ is determined by $\mathfrak{w}-\mathfrak{v}$ and $\mathfrak{v}\times\mathfrak{w}$; notation $\mathfrak{g}=(\mathfrak{v}, \mathfrak{w})=[\mathfrak{w}-\mathfrak{v}, \mathfrak{v}\times\mathfrak{w}]$. The vector $\mathfrak{p}$ determines the linear complex $\mathfrak{p}$ consisting of all $\mathfrak{g}$ with $\mathfrak{p}\circ\mathfrak{g}=0$. It is invariant under unimodular centro-affine transformations. An "A-figure" consists of (i) three linearly independent points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and three other points $\mathfrak{a}=a\mathfrak{a}$, $\mathfrak{b}=b\mathfrak{b}$, $\mathfrak{c}=c\mathfrak{c}$ such that $abc(1-abc)\neq 0$, (ii) the planes $\bar{\mathfrak{A}}\{\bar{\mathfrak{B}}, \bar{\mathfrak{C}}\}$ through $(\mathfrak{a}, \bar{\mathfrak{c}})\{(\mathfrak{b}, \bar{\mathfrak{a}}), (\mathfrak{c}, \bar{\mathfrak{b}})\}$ and parallel to $(\mathfrak{b}, \bar{\mathfrak{a}})\{(\mathfrak{c}, \bar{\mathfrak{b}}), (\mathfrak{a}, \bar{\mathfrak{c}})\}$ and the planes $\mathfrak{A}=a\bar{\mathfrak{A}}$, $\mathfrak{B}=b\bar{\mathfrak{B}}$, $\mathfrak{C}=c\bar{\mathfrak{C}}$. It is mapped onto itself by the correspondence $\mathfrak{a}\leftrightarrow\mathfrak{C}$, $\mathfrak{b}\leftrightarrow\mathfrak{A}$, $\mathfrak{c}\leftrightarrow\mathfrak{B}$, $\bar{\mathfrak{A}}\leftrightarrow\bar{\mathfrak{c}}$, $\bar{\mathfrak{B}}\leftrightarrow\bar{\mathfrak{a}}$, $\bar{\mathfrak{C}}\leftrightarrow\bar{\mathfrak{b}}$.

Let $\mathcal{A}$ be a two-parametric family of A-figures such that each of the six surfaces generated by their points $\mathfrak{a}$, $\bar{\mathfrak{a}}$, $\cdots$ has the corresponding planes $\mathfrak{C}$, $\bar{\mathfrak{B}}$, $\cdots$ as tangent planes. Thus corresponding points of the surfaces $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ {$\mathfrak{b}$ and $\bar{\mathfrak{b}}$, $\mathfrak{c}$ and $\bar{\mathfrak{c}}$} are collinear with O. Corresponding tangent planes of $\mathfrak{a}$ and $\bar{\mathfrak{b}}$ {$\mathfrak{b}$ and $\bar{\mathfrak{c}}$, $\mathfrak{c}$ and $\bar{\mathfrak{a}}$} are parallel. Finally, the straight lines through corresponding points of the surfaces $\mathfrak{a}$ and $\bar{\mathfrak{c}}$, etc. are the intersections of their tangent planes $\mathfrak{C}$ and $\bar{\mathfrak{A}}$, etc. There exists an imaginary quadric such that a duality with respect to it maps the tangent planes and points of $\mathcal{A}$ onto the points and tangent planes of six other surfaces with similar features. The resulting twelve surfaces form a Darboux family with the centre O [Darboux, Leçons sur la théorie générale des surfaces, vol. 4, Gauthier-Villars, Paris, 1896, pp. 48–86]. The surfaces $\mathfrak{a}$ and $\mathfrak{A}$, $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{A}}$, $\cdots$ are orthogonal, i.e. $d\mathfrak{a}\cdot d\mathfrak{A}=\cdots=0$.

Conversely, let $\mathfrak{p}=\mathfrak{p}(u, v)$ and $\bar{\mathfrak{p}}=\bar{\mathfrak{p}}(u, v)$ denote two twice continuously differentiable orthogonal surfaces such that $(\mathfrak{p}, \mathfrak{p}_u, \mathfrak{p}_v)\cdot(\bar{\mathfrak{p}}, \bar{\mathfrak{p}}_u, \bar{\mathfrak{p}}_v)\cdot\mathfrak{p}\bar{\mathfrak{p}}\neq 0$ and that the Pfaffians $\bar{\mathfrak{p}}d\mathfrak{p}$ and $\mathfrak{p}d\bar{\mathfrak{p}}$ are linearly independent. Then there exists a Darboux family with the centre O such that $\mathfrak{p}=\mathfrak{a}$, $\bar{\mathfrak{p}}=\mathfrak{A}$. In a given cartesian coordinate system it is unique and can be determined without integrations.

A congruence $\mathfrak{g}=\mathfrak{g}(u, v)$ of straight lines is called a W-congruence with the osculating complexes $\mathfrak{k}=\mathfrak{k}(u, v)$ if $\mathfrak{k}\circ\mathfrak{g}=\mathfrak{k}\circ d\mathfrak{g}=\mathfrak{k}\circ d^2\mathfrak{g}=0$. If the $\mathfrak{k}$'s do not degenerate, i.e. if $\mathfrak{k}\circ\mathfrak{k}\neq 0$, they can be normed such that $d\mathfrak{k}\circ d\mathfrak{k}=0$. Conversely, a family $\mathfrak{k}(u, v)$ with $d\mathfrak{k}\circ d\mathfrak{k}=0$ will in general consist of the osculating complexes of a W-congruence.

Suppose the family of the straight lines $(\bar{\mathfrak{b}}, \mathfrak{c})\subset\mathfrak{F}$ is actually two-parametric. Then it is a W-congruence with the osculating complexes $[\mathfrak{a}, \mathfrak{A}]$ and the focal surfaces $\mathfrak{c}$ and $\bar{\mathfrak{b}}$. If $\bar{\mathfrak{b}}$ has two distinct families of asymptotic lines, they correspond to those on $\mathfrak{c}$ and to conjugate nets on $\mathfrak{a}$, $\bar{\mathfrak{a}}$, $\mathfrak{b}$, and $\bar{\mathfrak{c}}$. Conversely, a W-congruence with non-degenerate osculating complexes can be imbedded into a family $\mathfrak{F}$ if it has two distinct focal surfaces and if all its focal points are finite. *P. Scherk* (Kingston, Ont.).

Backes, F. Sur les systèmes cycliques tels que l'axe du cercle générateur engendre une congruence de normales. Mathesis **65** (1956), 32–36.

Mathéev, A. Sur certaines questions de la géometrie différentielle des surface de l'espaces elliptique. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 2. **48** (1953/54), 77–86 (1954). (Bulgarian. French summary)

Le travail contient la généralisation des théorèmes de Meusnier et d'Euler pour les surfaces dans l'espace elliptique. Pour les surfaces réglées on écrit l'équation différentielle des courbes asymptotiques et aussi des courbes de courbure. Enfin on détermine les surfaces minima réglées. L'auteur emploit la méthode du repère mobile et des formes de Pfaff avec la désignation de M. Blaschke [Hamburger Math. Einzelschr. **34** (1942); MR **5**, 215].
F. Vyčichlo (Prague).

Muracchini, Luigi. Sulla applicabilità proiettiva delle superficie negli spazi a connessione proiettiva a tre dimensioni. Czechoslovak Math. J. **5(80)** (1955), 274–288. (Russian summary)

The author tackles again the notion of "projective applicability" of surfaces in a projectively connected manifold (introduced by this reviewer) using Cartan's methods. He proves that the projective connection can always be supposed without torsion or even normal, a fact which is absolutely obvious because that notion depends only on the paths of the connection, in accordance with the original definition, and not on its other characteristics. The final theorem (concerning the existence of a surface in a projective space applicable on a given surface of a projectively connected space) is an immediate consequence of the integrability conditions; but a more general statement, announced as probable, does not hold in general. *E. Bompiani* (Rome).

Tutaev, L. K. Curves with constant curvatures in the spaces: affine, projective, and Lorentz space of two and three dimensions. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **19** (1954), 3–26. (Russian)

In all these cases the differential equations of the curves are written up in the appropriate coordinates and integrated. As an example we take curves in two-dimensional Lorentz space, where we introduce $e_1^2=1$, $e_1e_2=0$, $e_2^2=-c^2$; the infinitesimal transformation of the canonical system of reference takes the form

$$\frac{d\bar{A}}{d\sigma}=e_1,\quad \frac{de_1}{d\sigma}=ke_2,\quad \frac{de_2}{d\sigma}=kc^2e_1,$$

from which follows for arbitrary k the equation

$$\bar{A}'''-k'k^{-1}\bar{A}''-k^2c^2\bar{A}'=0,$$

which integrated for $k=$const gives for the required curve $\bar{A}=\bar{A}(\sigma)$ a hyperbola $(cx_2+(kc)^{-1})^2-x_1^2=1/k^2c^2$.
D. J. Struik (Cambridge, Mass.).

Townsend, B. B. On extensive differentiation of extensors. Tensor (N.S.) **5** (1955), 101–110.

This is based upon a paper by H. V. Craig [Bull. Amer. Math. Soc. **53** (1947), 332–342; MR **8**, 491] in which higher order intrinsic derivatives of tensors are studied as contractions of extensors. A process for constructing extensors of range 0 to $r+1$ from absolute extensors of range 0 to r was introduced under the name of extensive differentiation. The author develops the process of extensive differentiation for extensors of more general types in this paper. He draws attention to the similarity of certain of his results with very general operators appearing in previous work by Kawaguchi [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **9** (1940), 1–152; MR **2**, 22].
E. T. Davies (Southampton).

Craig, Homer V. On certain linear extensor equations. II. Tensor (N.S.) **5** (1955), 77–84.

In this paper and the one to which this is a sequel [Tensor (N.S.) **4** (1954), 40–50; MR **16**, 626] the author is concerned with problems of mathematical physics and of the calculus of variations. In both domains there are certain key functions which appear in the problem and what he calls primary extensors associated with these functions. The author states as his general purpose that of testing the heuristic value of a new approach to problems in these two domains.

Whereas the usual procedure in physics consists of accepting the differential equations for a general problem as a starting point and constructing the Lagrangian function which has the property that its associated Euler-Lagrange equations are precisely the original set the author proceeds as follows: The final differential equations are not regarded as given but are to be derived from a set of data which consists of: (a) a postulated linear relationship in the primary extensors with initially undetermined coefficients; (b) a set of conditions for the determination of the coefficients.

In this paper he illustrates his method of approach by treating Maxwell's equations for points in free spaces, the flexible-string wave equation and a linear oscillator subject to a resisting force which is proportional to the velocity. *E. T. Davies* (Southampton).

Katsurada, Yoshie. On the functional tensor attached to an arc. Tensor (N.S.) **4** (1954), 16–27.

Vektoren und allgemeine Tensoren sind, in Mannigfaltigkeiten deren Grundelement der Punkt ist, Punktfunktionen. Verf. definiert nun diese Grössen für Räume, deren Grundelement durch Potenzreihen darstellbare Bogen sind. Sind $x^{(\alpha)i}$ die Koordinaten, die in einer erweiterten Punkttransformation von unendlichem Range vorkommen und $\bar{x}^{(\alpha)i}$ ihre Transformierten, dann sind die Komponenten des erweiterten kontravarianten Vektors

durch das Transformationsgesetz

$$\bar{v}^{\alpha\alpha}=\sum_{\beta=0}^{n}\frac{\partial\bar{x}^{(\alpha)\alpha}}{\partial x^{(\beta)\beta}}v^{\beta\beta}$$

bestimmt. Ein Bogen soll durch

$$x^i(t)=\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}x_0^{(\alpha)i}(t-t_0) \tag{1}$$

dargestellt sein. Ist die Potenzreihe

$$\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}v^{\alpha i}(u-u_0)^\alpha$$

in einem gewissen Intervall absolut konvergent, dann soll der zum Bogen (1) gehörige C-Vektor durch

$$C^{v(u)}\overset{\text{def}}{=}\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}v^{\alpha i}(u-u_0)^\alpha$$

definiert sein. Es wird nachgewiesen, dass die algebraischen Operationen für Vektoren, für C-Vektoren ihre Gültigkeit behalten.

Die zum Bogen (1) gehörigen C-Vektoren werden mit Hilfe des von Craig [Vector and tensor analysis, McGraw-Hill, New York, 1943; MR **5**, 77] eingeführten "crossed extensor" auf ganz entsprechende Weise definiert. Danach werden kovariante C-Vektoren und allgemeine C-Tensoren festgelegt. Mit Hilfe der vom Verf. eingeführten erweiterten Übertragungsparameter lässt sich ein erweitertes kovariantes Differential, sowie eine entsprechende Verallgemeinerung der Lieschen Ableitung definieren.

O. Varga (Debrecen).

Pastori, Maria. Un'insidia nell'uso di coordinate generali. Boll. Un. Mat. Ital. (3) **11** (1956), 72–79.

★ **Pailloux, H. Un aspect du calcul tensoriel.** Mémor. Sci. Math., no. 130. Gauthier-Villars, Paris, 1955. 74 pp. 1100 francs.

Le but principal de ce livre est de montrer que l'emploi du calcul fonctionnel, pour les systèmes matériels dont la position dépend de fonctions arbitraires, est beaucoup facilité par l'utilisation des notations du calcul tensoriel convenablement adaptées.

Dans le Chapitre I, intitulé „Calcul tensoriel et calcul fonctionnel", l'auteur définit d'abord l'espace $\mathcal{E}$ comme un ensemble dont les éléments sont des fonctions définies dans un domaine D, et il suppose qu'il existe des vecteurs de base dependant du point P variant dans D, tel qu'un vecteur $\mathbf{x}$ appartenant à l'espace affine $\mathcal{E}$ se mette d'une manière et d'une seule sous la forme suivante:

$$\mathbf{x}=\int_D x^P\mathbf{e}_P d\tau,$$

où $d\tau$ est l'élément de mesure du domaine D autour de P. On dit que x^P est une composante du vecteur $\mathbf{x}$. C'est une fonction du point P défini dans D. On peut constater le gros avantage qu'il y a à mettre le point P, dont depend cette fonction, en indice.

Après avoir dévelopé l'algèbre tensorielle, l'auteur définit la dérivée fonctionnelle de la manière suivante: Dans un petit domaine de masse $\Delta\tau$ entourant le point P donné, modifions la fonction x^P donnée d'une quantité constante Δx^P, x^P étant inchangée dans $D-\Delta\tau$. Soit Δf l'accroissement de la fonctionnelle. S'il existe une limite pour le rapport $\Delta f/\Delta\tau\Delta x^P$ lorsque Δx^P tend vers zéro, et qu'indépendemment, $\Delta\tau$ tend aussi vers zéro dans toutes ses dimensions et si la limite est unique, nous dirons que la limite est la dérivée fonctionnelle de f par rapport à la fonction x, au point P. On la note par $\partial f/\partial x^P$.

A l'aide de cette définition, l'auteur peut développer l'analyse tensorielle de la manière bien connue.

Dans le Chapitre II, „Premières variétés de l'espace $\mathcal{E}$", on dévelope la théorie des courbes. Chapitre III est consacré à la théorie d'intégration. L'auteur expose d'abord la notion d'intégrale dans un espace ordinaire et ensuite il passe au cas de l'intégration fonctionnelle. Dans le Chapitre IV on discute géodésiques. Ici aussi l'auteur discute d'abord les géodésiques d'une variété riemannienne et ensuite il passe au cas de géodésiques de nouvelles variétés.

Le chapitre V, intitulé „Mécanique analytique" est consacré aux problèmes tels que l'intégration d'Hamilton, fonctions-paramètres, systèmes à liaison indépendantes du temps. Le dernier chapitre contient quelques remarques au sujet de l'équation intégrale qui domine cette étude.

K. Yano (Tokyo).

See also: Montel, p. 998.

Riemannian Geometry, Connections

Clark, R. S. On conformal theory of a Riemannian space which admits an exterior form. J. London Math. Soc. **31** (1956), 83–88.

The author considers an n-dimensional Riemannian space $V(n, p)$ of class C^r ($r\geqq 3$) which carries, in addition to the positive definite metric tensor g_{ij}, an exterior p-form f with coefficients $f_{i_1i_2\cdots i_p}$ ($1<p\leqq n$) satisfying

$$f_{i_1\cdots i_{p-1}}{}^{r}f_{j_1\cdots j_{p-1}r}=(p-1)!\,g_{i_1j_1}.$$

When the exterior derivative (the covariant derivative with respect to g_{ij}) of the tensor $f_{i_1i_2\cdots i_p}$ vanishes, the $V(n, p)$ is called $S(n, p)$ ($K(n, p)$).

The author finds first a necessary and sufficient condition for $V(n, p)$ with $p<n$ to be equivalent to an $S(n, p)$ under a conformal transformation

$$g_{ij}'=e^{2\sigma}g_{ij},\quad f'_{i_1i_2\cdots i_p}=e^{p\sigma}f_{i_1i_2\cdots i_p}.$$

From this it follows that if $V(n, p)$ is conformal to an $S(n, n-1)$ it must be an $S(n, n-1)$ and that if two $S(n, p)$ are conformally related the function σ above must be a constant. He next finds a necessary and sufficient condition for $V(n, p)$ to be conformally equivalent to a $K(n, p)$. The semi-simple group manifold is a typical example of a $K(n, 3)$ and is denoted by $G(n, 3)$. The author finds a necessary and sufficient condition for $K(n, 3)$ to be conformally equivalent to a $G(n, 3)$. From this it follows that if two such local group manifolds are conformally related they must coincide.

The paper ends by global discussions which are analogous to those of W. J. Westlake [Proc. Cambridge Philos. Soc. **50** (1954), 16–19; MR **15**, 351]. *K. Yano* (Tokyo).

Upadhyay, M. D. Congruences of curves in a Riemannian space. II. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19** (1954), 19–22. (Turkish summary)

L'auteur envisage, dans un espace de Riemann V_m, $m-n$ congruences de courbes telles que, par chaque point d'une variété V_n ($n<m$) plongée dans V_n il passe une courbe de chaque congruence. Complétant les résultats d'un travail antérieur dans lequel sont données les définitions des foyers et des hypersurfaces focales d'une des congruences ci-dessus de V_m, il montre qeu

chaque courbe de la congruence est tangente en chaque foyer à l'hypersurface correspondante, et que toute hypersurface de la congruence touche l'hypersurface focale aux foyers de ses différentes courbes. *P. Vincensini.*

Nagai, Tamao. Some considerations on structures of simply transitive groups. Tensor (N.S.) **5** (1955), 91–94.

The author considers a simply transitive group of transformations, G_n, expressed by the symbols

$$X_a f = \xi_{(a)}{}^i \partial f / \partial x^i,$$

where the $\xi_{(a)}{}^i$ are n independent vectors. He presents a Riemannian space, R_n, in which the $\xi_{(a)}{}^i$ become components of unit vectors of an orthogonal ennuple, and finds relationships between the coefficients of structure of the G_n and the Ricci coefficients of rotation of the R_n. Using, in particular, results of T. K. Pan, who defined and studied γ_{ab}, the variation of one congruence of an orthogonal ennuple, $\xi_{(a)}$, with respect to another, $\xi_{(b)}$, and also the subvariation, [Canad. J. Math. **5** (1953), 519–523; MR **15**, 350], the author presents simple proofs for seven theorems, two of which follow: (3) If variations and subvariations of $\xi_{(p)}$, $p=1, \cdots, n-1$, with respect to $\xi_{(n)}$ are equal, then the subgroup expressed by the symbols $X_1 f, \cdots, X_{n-1} f$ is an invariant subgroup, and conversely. (7) If the $\xi_{(a)}{}^i$ are orthogonal normal congruences and $\gamma_{np}=0$, $p=1, \cdots, n-1$, then the system of invariant varieties for the invariant subgroup generated by the symbols $X_1 f, \cdots, X_{n-1} f$ is geodesically parallel. *A. Schwartz* (New York, N.Y.).

Ide, Saburo. On the connections in higher order spaces. II. Tensor (N.S.) **4** (1955), 135–140.

In einer früheren Arbeit [Tensor (N.S.) **3** (1954), 84–90; MR **15**, 989] hat Verf. in Räumen, deren Grundelement ein Linienelement höherer Ordnung ist, eine Übertragung eingeführt, die die Verallgemeinerung einer Wirtingerschen Übertragung [Trans. Cambridge Philos. Soc. **22** (1922), 439–448] ist. Anderseits hat A. Kawaguchi in den nach ihm benannten Räumen [Proc. Imp. Acad. Tokyo **13** (1937), 237–240] ebenfalls eine Übertragung definiert. In vorliegender Arbeit wird nun der Zusammenhang zwischen den beiden Übertragungsparametern festgestellt. Wegen der expliziten Formeln muss auf die Originalarbeit verwiesen werden. *O. Varga* (Debrecen).

Katsurada, Yoshie. On the theory of parallel paths. Tensor (N.S.) **4** (1954), 1–8.

Verf. hat in einer Reihe von früheren Arbeiten die Parallelübertragung von Bogen in solchen Räumen untersucht, deren Grundelement ein Linienelement höherer Ordnung ist. In der vorliegenden Arbeit werden notwendige und hinreichende Bedingungen dafür aufgestellt, dass die Bahnkurve eines affinzusammenhängenden Raumes L_n bei Parallelübertragung wieder in eine solche übergeht. Ist $R^m{}_{jkl}$ der Krümmungstensor des L_n, dann lassen sich diese Bedingungen durch

$$R^m{}_{(j|k|l)} = \frac{1}{n+1}(R^h{}_{h|kj}\delta_l{}^m + R^h{}_{l|kh}\delta_j{}^m)$$

bestimmen. Die Parallelübertragene einer Bahnkurve entlang ihrerselbst fällt mit der Ausgangsbahnkurve zusammen. Im Falle eines Riemannschen V_n ist die Übertragung für Geodätische nur in lokal ebenen Räumen möglich. Schliesslich wird gezeigt, dass die Bahnkurven des Differentialgleichungssystems

$$\frac{d^2x^i}{dt^2} + (\Gamma^i{}_{jk} + T^i{}_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt} = \varrho\frac{dx^i}{dt}$$

Parallelkurven sind, falls die kovariante Ableitung des symmetrischen Tensors $T^i{}_{jk}$ mit dem Krümmungstensor eine gewisse algebraische Relation erfüllt. *O. Varga.*

See also: Sasaki and Yano, p. 1004; Géhéniau et Debever, p. 1016.

Complex Manifolds

Sasaki, Shigeo; and Yano, Kentaro. Pseudo-analytic vectors on pseudo-Kählerian manifolds. Pacific J. Math. **5** (1955), 987–993.

A pseudo-Kählerian manifold is a Riemannian manifold M^{2n} of class C^r ($r \geq 2$) which admits a skew-symmetric tensor field I_{AB} of class C^{r-1}, with non-vanishing determinant, satisfying the conditions

$$I^A{}_B I^B{}_C = -\delta^A{}_C \quad (I^{AB} I_{BC} = -\delta^A{}_C), \quad I_{AB,C} = 0,$$

where $I^A{}_B = g^{AE} I_{EB}$, $I^{AB} = g^{AE} g^{BF} I_{EF}$, and the comma denotes covariant differentiation with respect to g_{AB}. Certain known theorems concerning the existence of analytic vector fields on Kähler manifolds are generalised to apply to pseudo-Kählerian manifolds. For example, when M^{2n} is a compact pseudo-Kählerian manifold, it is proved that if the Ricci tensor is positive definite then M^{2n} does not admit a pseudo-analytic covariant vector field other than the zero vector field. If the Ricci tensor is negative definite, then there exists no non-trivial pseudo-analytic contravariant vector field. *T. J. Willmore.*

Patterson, E. M. Riemann extensions which have Kähler metrics. Proc. Roy. Soc. Edinburgh. Sect. A. **64** (1954), 113–126.

A. G. Walker [Quart. J. Math. Oxford Ser. (2) **1** (1950), 69–79; MR **11**, 688] has proved that if a $2n$-dimensional Riemann space V_{2n} admits a parallel field π of null n-planes, then there are coordinates x^α such that the metric tensor $g_{\alpha\beta}$ has components given by

$$(1) \quad g_{ij} = g_{ij}(x^k), \quad g_{i+n,j} = \delta_{ij}, \quad g_{i,j+n} = \delta_{ij}, \quad g_{i+n,j+n} = 0,$$

where $\alpha, \beta, \cdots = 1, 2, \cdots, 2n$; $i, j, \cdots = 1, 2, \cdots, n$.

Suppose that the functions g_{ij} are given by

$$(2) \quad g_{ij} = X_{ij}{}^{pq}\xi_p\xi_q - 2\Gamma_{ij}{}^p\xi_p + c_{ij},$$

where $X_{ij}{}^{pq}(x^k)$, $\Gamma_{ij}{}^p(x^k)$ and $c_{ij}(x^k)$ are respectively tensor symmetric in p, q and i, j, a symmetric affine connection and a tensor symmetric in i, j, and $\xi_i = x^{i+n}$.

If $X_{ij}{}^{pq} = 0$, V_{2n} is called a Riemann extension of a space of affine connection [E. M. Patterson and A. G. Walker, ibid. **3** (1952), 19–28; MR **13**, 985]. If

$$X_{ij}{}^{pq} = \tfrac{1}{2}\theta(\delta_i{}^p\delta_j{}^q + \delta_j{}^p\delta_i{}^q - h^{pq}h_{ij})$$

and

$$\Gamma_{ij}{}^p = \left\{ {p \atop i\,j} \right\}_h - \tfrac{1}{2}(\delta_i{}^p\phi_j + \delta_j{}^p\phi_i - h_{ij}h^{pq}\phi_q),$$

where $\left\{ {p \atop i\,j} \right\}_h$ are Christoffel symbols formed with a non-degenerate tensor h_{ij}, then V_{2n} is a Riemann extension of a conformal space. If finally $X_{ij}{}^{pq} = \frac{1}{2}\theta(\delta_i{}^p\delta_j{}^q + \delta_j{}^p\delta_i{}^q)$, the V_{2n} is called a Riemann extension of a space of projective connection. In general a V_{2n} whose metric is given by (1) and (2) in some coordinate system is called a Riemann extension with respect to π.

The author first gives a necessary and sufficient condition for a V_{2n} admitting a parallel field of null n-planes π to be a Riemann extension with respect to π. If the components $g_{\alpha\beta}$ of the fundamental tensor of a V_{2n} are given in terms of coordinates y^α by

$$g_{ij}=0,\ g_{i,j+n}=g_{i+n,j},\ g_{i+n,j+n}=0$$

and

$$\partial g_{i+n,j}/\partial y^k=\partial g_{i+n,k}/\partial y^j,\ \partial g_{i,j+n}/\partial y^{k+n}=\partial g_{i,k+n}/\partial y^{j+n},$$

V_{2n} is called a Kähler space. A Riemann space V_{2n} is a Kähler space if and only if it admits two non-intersecting parallel fields of null n-plane. The author gives a necessary and sufficient condition for a space of metric (1) to admit a second parallel field π' of null n-planes such that $\pi\cap\pi'=0$. He also gives a necessary and sufficient condition for a Kähler space to be a Riemann extension with respect to π.

In the last section, the author proves that a Kähler space is symmetric if and only if it is a Riemann extension with respect to both parallel fields of null n-planes.

K. Yano (Tokyo).

Patterson, E. M. Symmetric Kähler spaces. J. London Math. Soc. **30** (1955), 286–291.

Following previous papers [J. London Math. Soc. **28** (1953), 260–269; MR **15**, 899; and the paper reviewed above], the author considers Kähler spaces with metrics which are not necessarily positive definite or Hermitian. He first proves that such a space is symmetric if and only if there exist coordinates $x^1, \cdots, x^n, \xi_1, \cdots, \xi_n$, and constants $X^{pq}{}_{ij}$ satisfying the condition

$$X_{pi}{}^{st}X_{jk}{}^{rp}+X_{pi}{}^{tr}X_{jk}{}^{sp}-X_{pj}{}^{rs}X_{ik}{}^{tp}-X_{pk}{}^{rs}X_{ij}{}^{tp}=0,$$

relative to which the metric takes the form

$$ds^2=X_{ij}{}^{pq}\xi_p\xi_q dx^i dx^j+2dx^i d\xi_i.$$

Then as an application he finds three canonical forms which give all possible irreducible, symmetric Kähler metrics for spaces of dimension four. *W. M. Boothby.*

Maruyama, Takaharu. On the angle between two planes in four dimensional space. Sci. Papers Fac. Engrg. Tokushima Univ. **2** (1950), no. 1, 1–4. (Japanese. English summary)

In three-dimensional space we define the angle between two planes as the angle between two normal lines with respect to one another. But in n-dimensional space one can not generally define the angle between two subspaces by such a method. Before entering into general considerations in n-dimensional space, the writer takes up the case of two planes in four-dimensional space.

From the author's summary.

Maruyama, Takaharu. On the geometry in a complex space. I. Sci. Papers Fac. Engrg. Tokushima Univ. **3** (1951), no. 1, 1–3. (Japanese. English summary)

If one considers a complex point in n-dimensional space, that is a point whose coordinates are complex numbers, as a real point in $2n$-dimensional space, one can construct a geometry in a complex space. As a particular case, one can represent the images of real and complex plane curves and straight lines as real surfaces and planes in four-dimensional space. At first some important properties of these planes, and toward the end some relations between the space under investigation and the unitary space are described. *Author's summary.*

Maruyama, Takaharu. Some properties on geometry in complex space. II. Sci. Papers Fac. Engrg. Tokushima Univ. **3** (1951), no. 2 (new no. 4), 1–5.

In the preceding paper [J. Sci. Gakugei Fac. Tokushima Univ. **1** (1950), 31–35; **2** (1952), i–v; MR **13**, 487; **14**, 902; see also the paper reviewed above] we have discussed some properties of holomorphic and isoclinic planes in four-dimensional space. In this paper we shall consider the transformations between isoclinic plane systems and holomorphic plane systems. To do that we first have to prove the transitivity of isoclinic relations; then we can determine several types of these transformations.

From the author's summary.

Maruyama, Takaharu. On the regularization of metrics in complex space. J. Gakugei Tokushima Nat. Sci. **3** (1953), 28–36.

Maruyama, Takaharu. On the regularization of metrics in complex space. Sci. Papers Fac. Engrg. Tokushima Univ. **1953.** no. 5, 1–6. (Japanese. English Summary)

In preceding papers we have discussed properties of geometry in complex space, but have not yet considered metric properties. In complex space, there are unitary metrics, and Hermite-Kähler metrics corresponding to Euclidean metrics and Riemannian metrics in real space, respectively. Since these metric functions are not regular, it is not convenient to introduce theories of regular functions into studies of geometry in complex space. Thus we wish to obtain regular functions whose norms have some geometrical meanings in complex space. We shall call this "regularization of metrics". In this paper we shall give some regular functions with respect to the regularization of metrics in two-dimensional complex space.

From the author's summary.

Algebraic Geometry

Vaňatová, Lada. Über eine Gattung von Gruppen der involutorischen ebenen Transformationen von Cremona. Časopis Pěst. Mat. **80** (1955), 152–171. (Czech. Russian and German summaries)

La position des six points du plan, quand il existe plus d'une seule involution symétrique, du cinquième degré de la première (resp. seconde) espèce, qui a des points pour éléments principaux, s'appelle la première (resp. seconde) position caractéristique [voir B. Bydžovský, Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1929**, no. 11]. L'auteur étude la question s'il existe le groupe de six points qui sont en même temps dans les deux positions caractéristiques. Le résultat qu'il a obtenu est le suivant: Six points du plan qui sont dans les deux positions caractéristiques peuvent être ordonnés en deux ou en six groupes des points qui ont la même propriété. Il n'existe aucun autre groupe formé par ces points. La première configuration donne le groupe G_8 d'ordre 8, la seconde mène au groupe G_{72} d'ordre 72. Ces groupes sont formées par les involutions et par les transformations cycliques du première ou du cinquième ordre. Ils existent les courbes du sixième degré qui sont invariantes par rapport aux transformations des groupes.

F. Vyčichlo (Prague).

Kubálková, Svatava. Zusammenhang der Hauptelemente einer symmetrischen Involution 5-ten Grades mit den Geraden einer kubischen Fläche. Časopis Pěst. Mat. **80** (1955), 172–190. (Czech. Russian and German summaries)

Les éléments principaux de l'involution symétrique

du 5ième degré peuvent être envisagés comme les figures planes des 27 droites de la surface S_3 du 3ième degré sans points doubles. Dans le mémoire on établie les conditions pour la surface S_3 dont les droites ont pour figures planes les éléments principaux formés par les six points du plan dans la première et seconde positions caractéristiques qui donnent les groupes G_8 resp. G_{72}. [Voir l'oeuvre analysée ci-dessus. *F. Vyčichlo* (Prague).

Spampinato, Nicolò. Curve e superficie osculatrici principali, a contatto tripunto o quadripunto, ad una falda di S_3 con l'origine in un punto o in una curva. Giorn. Mat. Battaglini (5) **3(83)** (1955), 41–53, 157–187 (1956).

Spampinato, Nicolò. Rappresentazione in S_5 di un fascio di curve e della serie lineare secata su una curva complessa prolungata sul campo biduale. Ricerca, Napoli **6** (1955), no. 2, 10–19.

Spampinato, Nicolò. La varietà dell' S_{11} determinata da una superficie algebrica dell' S_3 complesso. Ricerca, Napoli **6** (1955), no. 3, 9–16.

Godeaux, Lucien. Sur les droites des surfaces cubiques d'un système linéaire. Mathesis **65** (1956), 12–15.

Panella, Gianfranco. Le superficie di ordine $2n$ con otto punti n-pli situati sopra una quartica razionale o degenere. Rend. Mat. e Appl. (5) **13** (1955), 423–432.

On sait que 7 points d'une courbe gauche elliptique du 4ème ordre C^4 déterminent n^2 autres points de C^4, tels que chacun d'eux et les 7 points donnés sont 8 points n-ples pour des surfaces de l'ordre $2n$ appartenant à un système linéaire surabondant. L'A. considère ici le cas où la courbe C^4 est rationnelle ou dégénerée, tout en restant courbe base d'un faisceau de quadriques généralement irréductibles. Dans ce cas, pour que 8 points de C^4, qui ne soient pas les points de base d'un réseau de quadriques, peuvent être points n-ples pour une F^{2n} ne contenant pas C^4, il est nécessaire qu'une C^4 générale contenant 7 des 8 points soit elliptique. Cette première condition étant vérifiée, la condition pour que l'on puisse trouver sur C^4 un huitième point comme l'on veut, est que la série g_n^{n-1} découpée sur C^4 par les F^{2n} avec les 7 points n-ples possède un couple neutre. La courbe C^4 peut alors présenter seulement quatre formes différentes. L'A. considère aussi le cas où un des 7 points considérés sur C^4 est double pour C^4, et le cas où C^4 se décompose et F^{2n} contient une des ses composantes. *E. G. Togliatti.*

Burniat, Pol. Modèles simples de surfaces irrégulières à système canonique dégénéré. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 145–152.

Après quelques remarques générales sur les surfaces doubles, l'A. donne deux exemples de surfaces irrégulières dont le système canonique est composé avec des courbes fixes de genre p_g et des courbes de genre 2 ou 3 variables dans un faisceau. Le premier exemple est un cône cubique elliptique de genres $p_a=p_g-1$, $p^{(1)}=4p_g-3$; la partie variable du système canonique se compose de p_g-1 courbes doubles de genre 2. Le deuxième exemple est un cône double d'ordre 4 hyperelliptique avec les genres $p_a=p_g-2$, $p^{(1)}=8p_g-7$; la partie variable du système canonique se compose de p_g-1 courbes doubles de genre 3. *E. G. Togliatti* (Gênes).

Segre, Beniamino. Sur l'algébricité des courbes ayant un ordre relatif réel convenable. J. Math. Pures Appl. (9) **35** (1956), 43–54.

A set Γ of points in the real projective plane has order k_n relative to the curves C_n of order n in the plane if k_n is the exact upper bound of the number of points of Γ lying on a C_n which does not contain Γ. The author proves immediately the following. (a) If $k_n \leqq n$, Γ is contained in a line and $k_n=n$. (b) If $1<n<k_n\leqq 2n$, Γ lies on an irreducible conic and hence $k_n=2n$. (c) If $2n<k\leqq 3n$ and $n>3$, Γ is contained in an irreducible C_3 and hence $k_n=3n$ or $k_n=3n-1$.

The main interest lies in the reservation $n>3$ in (c), and the attempt to dispense with it by considering the cases $n=1$, 2 and 3. The cases $n=1$ and 2 are exceptional, the author showing that there exist sets Γ (which may even be real parts of algebraic curves) for which $k_1=3$ and $k_2=6$ simultaneously and yet Γ does not lie on a cubic. However, if $n=3$ and Γ consists of a finite number of homeomorphic images of a circle, and if $k_3=9$, then Γ consists of either (1) the real points of a cubic curve, or (2) the odd branch of such a curve (which contains a further oval) or (3) the points obtained from a real nodal cubic curve by suppressing the loop. *D. B. Scott.*

Monk, D. Jacobians of linear systems on an algebraic variety. Proc. Cambridge Philos. Soc. **52** (1956), 198–201.

Sur une variété algébrique V_d on considère k systèmes linéaires d'hypersurfaces S_1, S_2, $\cdots$, S_k, des dimensions r_1, r_2, $\cdots$, r_k dont la somme soit $h+1$; alors la variété Jacobienne de ces systèmes est donnée par la formule d'équivalence suivante:

$$(1) \qquad J_h[S_1\cdots S_k]\equiv K_h[\prod_{i=1}^{k}(1+S_i)^{r_i+1}];$$

où le symbole K_h appliqué à une variété V_t signifie le système canonique $K_h[V_t]$, ou bien la variété même V_t, ou bien 0, selon que $h<t$, $h=t$, $h>t$. La formule (1) a été démontrée par J. A. Todd dans le cas où les systèmes considérés sont des faisceaux ($r_1=\cdots=r_k=1$) et dans le cas où il se réduisent à un seul système ($k=1$) [Proc. London Math. Soc. (2) **45** (1939) 410–424; MR **1**, 84]. La même formule, dans le cas où $h=d-1$, se trouve dans un mémoire de F. Severi, qui a posé aussi le problème général [Ann. Mat. Pura Appl. (4) **32** (1951), 1–81, § 5; MR **13**, 979]. La démonstration que l'A. donne ici de la formule générale (1) procède par induction, en partant des résultats de J. A. Todd et de F. Severi. *E. G. Togliatti.*

Severi, Francesco. Complementi alla teoria della base. Ann. Mat. Pura Appl. (4) **40** (1955), 1–14.

Results on the theory of the base for surfaces obtained by the author in a number of papers [notably Atti Ist. Veneto Sci. Lett. Arti **75** (1916), 1121–1162; Accad. Italia Mem. Cl. Sci. Fis. Mat. Nat. **5** (1934), 239–283] are extended to varieties V_r of dimension $r\geqq 3$. For hypersurfaces of V_r the author establishes the identity of pseudohomology, algebraic pseudo-equivalence and numerical equivalence. He gives various tests for algebraic equivalence of hypersurfaces, including one in terms of equivalence of sections by a general surface of V_r, and proves that all $(2r-2)$-dimensional zero-divisors are algebraic. The final section sketches some lines along which the author is working to extend the theory of the base fully to subvarieties of lower dimension.

D. B. Scott (London).

Vesentini, Edoardo. Un'osservazione sul teorema dell'appartenenza. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 38–43.

Segre's "teorema dell'appartenenza" [Ann. Mat. Pura Appl. (4) **35** (1953), 1–127; MR **15**, 822–1140] asserts, in the notation of the paper cited, $\{P_V\}=(\{P_M\}\cdot\{M_V\})_M$. The proof given by Segre supposes that we can find a non-singular variety N such that $M \subset N \subset V$. In this paper the theorem is derived differently, the main tool being a new covariant succession $\{v^P\}$ of systems of equivalence, whose definition is too complicated to be given explicitly here, and which is eventually proved to be the inverse $\{\tilde{P}_V\}$ of Segre's system $\{P_V\}$. *D. B. Scott* (London).

Northcott, D. G. Abstract dilatations and infinitely near points. Proc. Cambridge Philos. Soc. **52** (1956), 178–197.

Soient Q un anneau local régulier de dimension 2 (d'égales ou inégales caractéristiques, peu importe), $\mathfrak{m}$ son idéal maximal. Un idéal premier principal $\mathfrak{p}$ de l'anneau gradué associé $G(Q)=\sum_n \mathfrak{m}^n/\mathfrak{m}^{n+1}$ (qui est un anneau de polynômes à deux variables sur $Q/\mathfrak{m}$) est appelé un point proche du premier ordre. On note alors $Q_\mathfrak{p}$ l'anneau de fractions Q_S où S est l'ensemble (multiplicativement stable) des éléments de Q dont la forme initiale dans $G(Q)$ n'est pas dans $\mathfrak{p}$. L'anneau $Q_\mathfrak{p}$ est un anneau local régulier de dimension 2, $\mathfrak{m}Q_\mathfrak{p}$ est un idéal premier principal, et $Q_\mathfrak{p}/\mathfrak{m}Q_\mathfrak{p}$ est l'anneau d'une valuation discrète; notons $\mathfrak{m}_\mathfrak{p}$ l'idéal maximal de $Q_\mathfrak{p}$. Soit $\mathfrak{q}$ un idéal de dimension l de Q; un point proche du premier ordre $\mathfrak{p}$ est dit „sur $\mathfrak{q}$" si $\mathfrak{p}$ contient l'idéal directeur $G(\mathfrak{q})$ de $\mathfrak{q}$ dans $G(Q)$; s'il en est ainsi et si $\mathfrak{q}$ est premier, $Q_\mathfrak{p}$ est un sous-anneau de l'anneau de fractions $Q_\mathfrak{q}$, et on pose $\mathfrak{q}_\mathfrak{p}=\mathfrak{q}Q_\mathfrak{q}\cap Q_\mathfrak{p}$. Soit f un élément non inversible de Q; dans $Q_\mathfrak{p}$ on a la relation $(f)=(\mathfrak{m}Q_\mathfrak{p})^n(f_\mathfrak{p})$ où l'idéal principal $(f_\mathfrak{p})$ est déterminé de façon unique par f. On démontre la formule suivante, où $\mathfrak{q}$ désigne un idéal premier de dimension 1 de Q, f un élément non inversible de Q tel que $(\mathfrak{q}, f)$ soit primaire pour $\mathfrak{m}$, et L la multiplicité d'un anneau local, et L la longueur d'un idéal primaire:

$$L(\mathfrak{q}, f)=e(Q/\mathfrak{q})e(Q/(f))+\sum_\mathfrak{p} L(f_\mathfrak{p}, \mathfrak{q}_\mathfrak{p})[Q_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}:Q/\mathfrak{m}].$$

Par décomposition primaire de (g) on en déduit une formule analogue où $\mathfrak{q}$ est remplacé par un idéal principal (g) de Q tel que (f, g) soit primaire pour $\mathfrak{m}$. On a aussi $e(Q/(f))=\sum_\mathfrak{p} L(f_\mathfrak{p}, \mathfrak{m}Q_\mathfrak{p})[Q_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}:Q/\mathfrak{m}]$. Par applications répétées on définit les points proches d'ordre supérieur. On déduit des formules ci-dessus une généralisation de la formule de Noether pour les multiplicités d'intersection, et une généralisation des conditions pour qu'un système d'entiers soient les multiplicités d'une courbe en une suite finie donnée de points proches consécutifs. *P. Samuel.*

See also: Hasse, p. 947; Lamprecht, p. 947; Brauner, p. 997; Montel, p. 998.

NUMERICAL ANALYSIS

★**Kopal, Zdeněk. Numerical analysis. With emphasis on the application of numerical techniques to problems of infinitesimal calculus in single variable.** John Wiley & Sons, Inc., New York, 1955. xiv+556 pp. $12.00.

The author presents a complete development of the classical subjects of numerical analysis listed in the chapter headings: Polynomial interpolation; Numerical differentiation; Integration of ordinary differential equations; Boundary-value problems: algebraic methods; Boundary-value problems: variational, iterative, and other methods; Mechanical quadratures; Numerical solution of integral and integro-differential equations. In addition, he gives at the end of each chapter a splendid account of the history of the subject with detailed bibliographical references. Problem sets are listed at the end of each chapter.

The author precedes the main body of the text with an historical and philosophical introductory chapter; while he concludes with five appendices which contain useful tables and treatments of special topics.

"The aim of the present work has been twofold: to provide an advanced undergraduate textbook, as well as a research handbook for certain branches of numerical analysis." *E. Isaacson* (New York, N.Y.).

Householder, Alston S. Bibliography on numerical analysis. J. Assoc. Comput. Mach. **3** (1956), 85–100.

This bibliography appeared earlier as a report [Oak Ridge National Laboratory, Rep. ORNL **1897** (1955); MR **16**, 1053].

Gorn, S. On the study of computational errors. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **816** (1952), 42 pp.

The author studies the influence of errors in input data and accumulating round-off error in numerical computations, both hand and machine. Four distinct basic cases, on which the efficiency of high-speed digital computation depends, are analyzed. These are (1) the influence of round-off on the elementary operations, (2) scaling and error bounds in the evaluation of a polynomial, (3) error analysis of Newton's method for obtaining square roots, and (4) error in a real root of a polynomial. The analysis is quite detailed, and many quantitative results are given. *W. S. Loud.*

Gorn, S.; and Juncosa, M. L. Some estimates of confidence limits of errors in evaluating polynomials and adding numbers. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. no. **642** (1952), 16 pp.

From the authors' abstract: Upper bounds on confidence limits corresponding to the $p\%$ level on errors in evaluating polynomials and adding numbers are given under the assumption that the errors are uniformly distributed and mutually uncorrelated. *M. Muller.*

Vernotte, Pierre. Les applications scientifiques de la notion de régularité. Publ. Sci. Tech. Ministère de l'Air, Paris no. **307** (1956), ii+84 pp. 750 francs.

Pogrebysskiĭ, I. B. On computation of integrals of rapidly oscillating functions and on exponential interpolation. Ukrain. Mat. Ž. **7** (1955), 291–294. (Russian)

The author discusses approximation of a function $f(y)$ over a finite interval by exponential interpolation, that is, by a sum of the form (1) $\sum_{k=1}^{n} A_k \exp(\beta_k y)$. The technique is of value in evaluation of integrals of the form $\int_a^b f(y) \exp(i\lambda y)dy$ when λ is real and large. Assuming $f(y)$ to be given at $2n$ equally spaced values of y, the author

shows how the coefficients A_k and β_k can be found so that (1) is equal to $f(y)$ for the given values. It is necessary to solve a system of n equations of first degree, and one equation of nth degree. If $f(y)$ is given at $s>2n$ equally spaced values, it is necessary to use the method of least squares or a similar technique. No quantitative discussion of error is given. The technique of exponential interpolation is mentioned briefly in Whittaker and Robinson, "Calculus of observations" [3rd ed., Blackie London-Glasgow, 1940]. *W. S. Loud.*

Luke, Yudell L. Evaluation of an integral arising in numerical integration near a logarithmic singularity. Math. Tables Aids Comput. **10** (1956), 14–21.

L'auteur donne une représentation de:

$$\int_0^{rh} x^{-1}dx \int_0^x f(u)du$$

par une formule approchée du type

$$(rh/D_n)\sum_{j=0}^{n}\gamma_{jr}^{(n)}f_j+R_{nr}$$

exacte pour les polynomes de degré n. Les f_j sont les valeurs de $f(u)$ aux points $u=jh$. Il donne les coefficients j pour $n=1(1)10$, $r=1(1)(n+1)$. Les formules écrites sont définies sauf si $r=n+1$ avec n impair. Les formules proposées sont comparées à celles de Kaplan [J. Math. Phys. **31** (1952), 1–28; MR **13**, 782.]. *J. Kuntzmann.*

Eminton, E.; and Lord, W. T. Note on the numerical evaluation of the wave drag of smooth slender bodies using optimum area distributions for minimum wave drag. J. Roy. Aero. Soc. **60** (1956), 61–63.

The integral

$$I(S)=-\frac{1}{2\pi}\int_0^1\int_0^1 S''(x)S''(y)\log|x-y|\,dxdy$$

is approximated from below, by replacing $S(x)$ by the piecewise smooth function $S_n(x)$. Here $S_n(x)$ is defined by the following properties. (a) $S_n(x)$ agrees with $S(x)$ at 0, 1 and the n distinct points $\{x_i\}$. (b) $I(S_n)$ is a minimum for all such piecewise smooth functions which have the same values at the points $\{x_i\}$. The main reason for using $S_n(x)$ is that $I(S)_n$ can be trivially evaluated. In spite of the fact that $S_n''(x_i)$ is infinite, the authors find that $I(S_n)$ approximates $I(S)$ satisfactorily for a wide class of functions $S(x)$, with $n=19$. $S(x)$ is the cross-sectional area of a smooth slender body for $0\leq x\leq 1$. I is equal to D/q, where D is the wave drag and q is the kinetic pressure.

E. Isaacson (New York, N.Y.).

★**Hasselmeier, H. Darstellung der Differentialquotienten aus einem verketteten Gleichungssystem mit einer oder mehreren unabhängigen Variablen.** Jenaer Jahrbuch 1955, 1. Teil, pp. 126–145 (1 plate). Gustav Fischer Verlag, Jena, 1955.

Let $P_i(y_i, \cdots, y_r, x_1, \cdots x_e)=0$ $(i=1, \cdots, r)$ be r linearly independent, sufficiently differentiable equations in r dependent and e independent variables. The author sets up a general recursive scheme for reckoning all partial derivatives $\partial^\nu y_i/\partial x_\mu^\nu$. The scheme is said to be broken into simple operations permitting easy checks. Fundamental determinantal expressions are isolated which simplify the later calculations. The case $e=1$ is exhibited in great detail, with a table of coefficients for getting $d^\nu y_1/dx_1^\nu$ $(\nu=1, \cdots, 9)$. The paper is not concerned with how well the formulas are adapted to numerical computation.

G. E. Forsythe (Los Angeles, Calif.).

Langefors, Börje. On the practical solution of linear equations. Svenska Aeroplan A. B. Tech. Note No. **35** (1955), 24 pp.

The use of inverse matrices is compared with the using of results from elimination work for obtaining solutions to systems of equations which have previously been solved for other right-hand members. Some procedures for approximately solving systems of equations and for iterative solutions are further discussed and a new approach to the solution of such equations, which may be established by the aid of matrix transformation, is suggested. *Author's summary.*

Langefors, Börje. Ill-conditioned matrices. Svenska Aeroplan A. B. Tech. Note No. **22** (1953), 18 pp.

The errors occuring in the solutions of systems of equations as a consequence of limited accuracy in the coefficients are discussed. Applications to inversion of ill-conditioned matrices, to numerical computations and to analog computers are given. *Author's summary.*

Madić, Petar. L'étude de solubilité des systèmes des équations algébriques linéaires. Bull. Inst. Nuclear Sci. "Boris Kidrich" no. **18** (1952), 13–15.

To solve systems (*) $\sum_{j=1}^{n}a_{ij}x_j=b_i$ by analog devices the author considers the method of least squares, in which components x_p are successively altered to minimize $\sum_{i=1}^{n}(\sum_{j=1}^{n}a_{ij}x_j-b_i)^2$. Let $\alpha_{ij}=a_{ij}(\sum_{i=1}^{n}a_{ij}^2)^{-\frac{1}{2}}$. It is observed that, if $\sum_{i=1}^{n}\alpha_{ip}\alpha_{iq}=\gamma_{pq}=0$ (all $p\neq q$), the solution of (*) is obtained after one alteration of each of the x_p. The author therefore introduces the $n(n-1)/2$ numbers γ_{pq} as measures of the "degree of solvability" of (*). If all $|\gamma_{pq}|<0.7$, the system (*) is said to be easily solved.

Reviewer's note: Thus one might adjoin

$$\Gamma=\{1-\max|\gamma_{ij}|\}^{-1}\ (\geqq 1)$$

to the various matrix "condition numbers" listed by Turing [Quart. J. Mech. Appl. Math. **1** (1948), 287–308; MR **10**, 405] and Todd [Proc. Cambridge Philos. Soc. **46** (1950), 116–118; MR **11**, 403]. *G. E. Forsythe.*

★**Lotkin, M.; and Remage, R. Matrix inversion by partitioning.** Proceedings of the Association for Computing Machinery, Toronto, 1952, pp. 36–41. Sauls Lithograph Co. (for the Association for Computing Machinery), Washington, D.C., 1953.

The inversion of a matrix by calculating in sequence the inverses of a chain of principal minors is here studied for positive definite symmetric matrices. A fully detailed analysis is made of the scaling needed at the beginning and in the course of the computation to allow fixed point operations. The number of multiplications and additions required is found to be $\frac{1}{2}(n-1)n(n+2)$ and $\frac{1}{2}(n-1)n(n+1)$, respectively. No estimate is made of the number of examinations and shifts needed for scaling but the desirability of double precision accumulation of inner products at certain places is noted. A bound for the rounding error introduced in going from one minor to the next is quoted from another paper by the same authors [Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **823** (1952); cf. also Ann. Math. Statist. **24** (1953), 428–439; MR **15**, 66]. This bound (the necessary definitions make it too long to quote) is said to be sufficiently close to have been of practical use.

W. Givens (Knoxville, Tenn.).

Fiedler, Miroslav; und Pták, Vlastimil. Über die Konvergenz des verallgemeinerten Seidelschen Verfahrens zur Lösung von Systemen linearer Gleichungen. Math. Nachr. **15** (1956), 31–38.

The relations $A<B$, $A\leqq B$, etc. among square matrices here mean that $a_{ij}<b_{ij}$, $a_{ij}\leqq a_{ij}$, etc. for all $i, j=1, \cdots, n$. Let E be the identity. Suppose $A^T=A\geqq 0$, while $E-A$ is positive definite. For each matrix B with $0\leqq B\leqq A$, the authors define a "generalized Seidel" iteration for solving the linear system $x=Ax+y$. Their method is to determine a vector sequence $x^k\to(E-A)^{-1}y$ by the relations: (*) $(E-B)x^{k+1}=(A-B)x^k+y$. The convergence-factor of (*) is called $\lambda(B)$; it is the modulus λ of the dominant eigenvalue of $(E-B)^{-1}(A-B)$.

The principal result is that $0\leqq B_1\leqq B_2\leqq A$ implies $\lambda(B_2)\leqq\lambda(B_1)<1$. They also touch on the dependence of $\lambda(B)$ on A. As applications of their result the authors consider various forms of group-relaxation. For example, let α map the integers $1\leqq i\leqq n$ onto the integers $1, \cdots, r$ $(r\leqq n)$. Let $b_{ik}=a_{ik}$, for $\alpha(i)\geqq\alpha(k)$, while $b_{ik}=0$ otherwise. Then, if α_2 is a "refinement" of α_1, it follows that $\lambda(B_{\alpha_1})\leqq\lambda(B_{\alpha_2})$. *G. E. Forsythe* (Los Angeles, Calif.).

Pogrebysskiĭ, I. B. Graphical solution of systems of linear algebraic equations. Ukrain. Mat. Ž. **7** (1955), 419–422. (Russian)

The methods of descriptive geometry are applied, but recommended only for systems of order 5 or less. The method is illustrated merely by a system of order 3 and one of order 4, and is not easily described in a few words. For order 3, in the plane of the graph one axis represents x_1 or x_3 interchangeably, the other x_2. Hence the paper represents simultaneously two of the coordinate planes (four for a 4th order system). One proceeds by plotting sections and projections in appropriate sequence, obtaining ultimately the projections of the solution upon the two coordinate planes. *A. S. Householder.*

Falk, Sigurd. Neue Verfahren zur direkten Lösung algebraischer Eigenwertprobleme. Abh. Braunschweig. Wiss. Ges. **6** (1954), 166–194.

The problem of the title is finding numbers k and vectors $\mathfrak{x}$ such that $(\mathfrak{A}-k\mathfrak{B})\mathfrak{x}=0$, where $\mathfrak{A}$, $\mathfrak{B}$ are arbitrary real or complex square matrices of order n, singular or not. There is first a review of several direct methods, mostly classified by the type of transformation matrices $\mathfrak{L}$, $\mathfrak{R}$ such that the problem becomes (*) $\mathfrak{A}^*\mathfrak{y}=k\mathfrak{B}^*\mathfrak{y}$, where $\mathfrak{A}^*=\mathfrak{L}\mathfrak{A}\mathfrak{R}$, $\mathfrak{B}^*=\mathfrak{L}\mathfrak{B}\mathfrak{R}$. The author introduces methods I, II, III. In I, $\mathfrak{A}^*$ is a "Hessenberg matrix" (i.e., $a_{ij}^*=0$ for $i>j+1$), while $\mathfrak{B}^*$ is upper triangular; $\mathfrak{L}$ and $\mathfrak{R}$ are independent of k. It is then fairly easy to evaluate $\phi^*(k)=|\mathfrak{A}^*-k\mathfrak{B}^*|$ by a recurrence and thus determine the roots k. One then gets the y from (*), and $\mathfrak{x}=\mathfrak{R}\mathfrak{y}$. In II, one precedes I by row operations making $\mathfrak{B}$ lower triangular, then $\mathfrak{B}^*$ becomes $\mathfrak{E}$, and computing $\mathfrak{L}$ and $\mathfrak{R}$ is easier. When $\mathfrak{B}=\mathfrak{E}$, method I reduces to that of Hessenberg [R. Zurmühl, Matrizen, Springer, Berlin-Göttingen-Heidelberg, 1950; MR **12**, 73].

In III, $\mathfrak{L}$ and $\mathfrak{R}$ are functions of k. It is a generalization for $\mathfrak{B}\neq\mathfrak{E}$ of Weber's method for $\mathfrak{B}=\mathfrak{E}$ [Rech. Aéro. no. 10 (1949), 57–60; MR **11**, 266]. III has the advantage over I of also applying to problems of form

$$(\mathfrak{A}k^2+\mathfrak{B}k+\mathfrak{C})\mathfrak{x}=\mathfrak{f}.$$

There are numerical examples of orders 3 or 4 for each method. Comparative counts show that III requires fewer multiplications than other direct methods.

G. E. Forsythe (Los Angeles, Calif.).

Voroncov, B. D. On a method of solving equations. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 187–190. (Russian)

The author regards this as a generalization of Schröder's algorism [Math. Ann. **2** (1870), 317–365]. It is in fact a special case, and has been well explored by Richmond [J. London Math. Soc. **19** (1944), 31–38; MR **6**, 133]; Hamilton [Duke Math. J. **13** (1946), 113–121; Amer. Math. Monthly **57** (1950), 517–522; MR **7**, 488; **12**, 537]; and this reviewer [Principles of numerical analysis, McGraw-Hill, New York, 1953; MR **15**, 470]. Only Schröder's paper is cited by the author. *A.S. Householder.*

Calvo Carbonell, Carlos. Studies on the numerical solution of equations of 3rd, 4th, and 5th degree. Gac. Mat., Madrid (1) **7** (1955), 14–26, 60–78, 109–124. (Spanish)

By a series of transformations, a general cubic equation with real coefficients $a_0x^3+a_1x^2+a_2x+a_3=0$ is reduced to $X^3\pm X+k=0$, where k is given explicitly in terms of the a_i. The roots of the reduced equation are then rapidly calculated with a table of cubes and a table of roots given here for a succession of values of k. Interpolation in the method is discussed in detail. Equations of fourth degree are reduced to the preceding case by the Euler resolvent. The method is extended to equations of degree n.

E. Frank (Chicago, Ill.).

Kuntzmann, J., rédacteur. Formules de dérivation approchée au moyen de points équidistants. Société d'Électronique et d'Automatisme, Courbevoie, Rep. Tech. **1 373/1** (1954), 47 pp.

A set of tables of coefficients are presented which may be used to calculate the n derivatives of a tabulated function at any midpoint, given $n+1$ equally spaced values of the function. Tables are given for $n=2, 3, 4, \cdots, 10$. *H. Polachek* (Carderock, Md.).

Rowe, Paul P. Difference approximations to partial derivatives for uneven spacings in the network. Trans. Amer. Geophys. Union **36** (1955), 995–1008.

Three methods, based respectively on Taylor's series, operator calculus, and "stencil mathematics" are used to develop various formulae, with error terms, for the finite-difference representation of certain elliptic partial differential equations, using unequal intervals in the two-dimensional grid spacing. All the methods are well known, though not all the results have been published previously.

L. Fox (Teddington).

Blatter, Christian. Eine Modifizierung der Simpsonschen Regel. Elem. Math. **11** (1956), 56–59.

Zanaboni, Osvaldo. Procedimento di risoluzione approssimata delle equazioni differenziali ordinarie. Riv. Mat. Univ. Parma **6** (1955), 89–110.

Die Differentialgleichung $L(y)=0$ wird zerlegt in $M(y)=N(y)$, der Differentialausdruck $N(y)$ sei von niedrigerer Ordnung als $M(y)$. Diese Zerlegung wird so gewählt, dass $M(y)=f(x)$ formal leicht integrierbar ist. Man wähle nun in einer dem Problem und den Rand- bzw. Anfangsbedingungen angepassten Weise für die gesuchte Funktion einen Ansatz $y=\varrho_0(x)$, der einen oder mehrere Freiwerte enthält. Diesen Ausdruck setze man in $N(y)$ ein. Sei $N(\varrho_0)=f(x)$. Durch Auflösung von $M(y)=f(x)$, unter Berücksichtigung von Rand- bzw. Anfangsbedingungen, findet man $y=\varrho_1(x)$. Die Freiwerte werden

durch Gleichsetzen der Werte an einer bzw. mehreren Stellen des Integrationsintervalles bestimmt. Das Verfahren wird an sechs Beispielen ausführlich erläutert. Im Verhältnis zur geringen Rechenarbeit ergibt sich eine recht befriedigende Genauigkeit. *P. Funk* (Wien).

Kunztmann, Jean. Remarques sur la méthode de Runge-Kutta. C. R. Acad. Sci. Paris **242** (1956), 2221–2223.

A step-by-step method for numerical solution of $y'=Y(y, t)$ which determines y_1, an approximate value of $y(t_0+h)$, starting from y_0, an approximate or given value of $y(t_0)$, and using intermediate values $y_{0,t}$ and $Y_{0,t}$, is called normal of rank n in case (1) it uses n intermediate points $y_{0,t}$, $Y_{0,t}$, (2) the error is of order h^{n+1}, and (3) all intermediate points have an ordinate with error of order h^n. The author states that in the family of methods containing the classical Runge-Kutta method, the methods of rank 1 and 2 are normal, and those of ranks 3 and 4 are not. Similar properties exist for the Runge-Kutta formulas for equations of second order.

W. S. Loud (Cambridge, Mass.).

Gautschi, Walter. Über den Fehler des Runge-Kutta-Verfahrens für die numerische Integration gewöhnlicher Differentialgleichungen n-ter Ordnung. Z. Angew. Math. Phys. **6** (1955), 456–461.

"For the numerical solution of an ordinary differential equation of the nth order, $y^{(n)}(x)=f(x, y, y' \cdots, y^{(n-1)})$, with appropriate initial conditions, Zurmühl has developed general Runge-Kutta formulas which integrate the equation directly without separating it into a system of n equations of first order. In the present paper some estimations of the errors of this Runge-Kutta method are given, which are valid for any value of n." The estimates of error are based on a priori bounds of the first four partial derivatives of $f(x, y, y', \cdots, y^{(n-1)})$ in a neighborhood of the initial values. *E. Isaacson* (New York, N.Y.).

Grossman, D. P. On the calculation of the characteristic values of Laplace's operator by the method of nets. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 595–597. (Russian)

The characteristic values ϱ_t are approximated by $\varrho_t^{(h)}$ and expressed in the form $\varrho_t^{(h)}=4(\lambda_t^{(h)}-1)/h^2$ with the net established in the usual way. Then the quantities $16(\lambda_t^{(h)})^2$ satisfy an algebraic equation with integral coefficients which, when the number of points is not too great, are obtainable from simple geometric considerations.

A. S. Householder (Oak Ridge, Tenn.).

Mitchell, A. R. Round-off errors in implicit finite difference methods. Quart. J. Mech. Appl. Math. **9** (1956), 111–121.

"Symmetrical and asymmetrical implicit finite difference replacements involving a variable parameter a and a variable mesh ratio s are considered for the heat conduction and wave equations, and expressions obtained for the round-off errors. It is found that the stable backward-difference replacements, four-point for the heat conduction equation and five-point for the wave equation, give rise to minimum round-off errors." *E. Isaacson.*

Cuénod, Michel. Application du calcul à l'aide de suites à l'étude de phénomènes de propagation. Bull. Tech. Suisse Romande **82** (1956), 19–27.

This paper discusses the use of 'time-series' in problems of propagation with reflection, such as water-hammer in pipes and surges on transmission lines. It is essentially identical with material in the author's published thesis [Ecole Polytech. Fédérale, Zürich, 1955; MR **17**, 535].

A. Tustin (London).

Nagai, Mashige. Graphical solution of linear heat flow with radiation. J. Phys. Soc. Japan **11** (1956), 329–330.

Položiĭ, G. N. Conformal mapping of simply connected and doubly connected regions and the determination of the Christoffel-Schwarz constants by means of a mathematical apparatus. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 15–18. (Russian)

A simple electrical device, based on the measurement of the resistance of conductors, is proposed for the determination of the constants appearing in the Christoffel-Schwarz formula for the conformal mapping of a circle on a given polygonal domain. An analogous procedure is indicated for determining whether two given doubly-connected domains are conformally equivalent.

W. Seidel (Notre Dame, Ind.).

Bahvalov, S. V. On nomograms with binary fields. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **21** (1954), 73–80. (Russian)

The components of the three-dimensional vectors $\mathbf{C}$, $\mathbf{A}$ and $\mathbf{B}$ are, respectively, functions of l variables u_α, m variables v_β and n variables w_γ while the components of $\mathbf{P}$ are functions of the v_β and w_γ. $\frac{1}{2}(m+n)(m+n+1)$ conditions, expressed as the vanishing and non-vanishing of third-order determinants, are shown to be necessary and sufficient for a function given in the form $\mathbf{C}\cdot\mathbf{P}$ to be expressible as $\mathbf{C}\cdot\mathbf{A}\times\mathbf{B}\sigma$, where σ is a scalar function of the v_β and w_γ. The conditions are equivalent to: $\mathbf{P}$, $\partial\mathbf{P}/\partial v_\beta$ and $\partial^2\mathbf{P}/\partial v_\beta^2$ are all coplanar; $\mathbf{P}$, $\partial\mathbf{P}/\partial w_\beta$ and $\partial^2\mathbf{P}/\partial w_\gamma^2$ are all coplanar and no $\partial\mathbf{P}/\partial v_\beta$ is coplanar with $\mathbf{P}$ and a $\partial\mathbf{P}/\partial w_\gamma$. O. D. Kellog [Z. Math. Phys. **63** (1915), 159–173] dealt more extensively with the case $l=m=n=1$. The above concise vector formulation, suggested by related work of I. A. Vil'ner [Dokl. Akad. Nauk SSSR (N.S.) **90** (1953), 5–8; MR **16**, 754], is due to the reviewer.

R. Church (Monterey, Calif.).

Bahvalov, S. V. Nomographing of tables. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **21** (1954), 81–82. (Russian)

If nm values of $f(x, y)$ in a double-entry table are such that $f(x_i, y_j)<f(x_{i+1}, y_j)$ and $f(x_n, x_j)<f(x_1, y_{j+1})$, it is shown that they can be used to construct a nomogram with parallel scales, uniform for x and y, that will reproduce them exactly for $x=x_i$ and $y=y_j$. The scale for $z=f(x, y)$ is determined by the Lagrange interpolation formula. *R. Church* (Monterey, Calif.).

Kreĭnes, M. A.; and Aĭzenštat, N. D. On nomographing with accuracy up to infinitesimals of higher order. Mat. Sb. N.S. **37(79)** (1955), 337–352. (Russian)

$z=f(x, y)$ is said to be nomographable with accuracy up to infinitesimals of kth order in the neighborhood of the "ordinary" point x_0, y_0, z_0 if, briefly, $f(x, y)-N(x, y)=O(\varrho^k)$, where $\varrho=[(x-x_0)^2+(y-y_0)^2]^{\frac{1}{2}}$ and $z=N(x, y)$ is equivalent, in the neighborhood of x_0, y_0, z_0, to the vanishing of $\Delta(x, y, z)$, a Massau determinant. Attention can be restricted to $x_0=y_0=z_0=0$ and it is shown that transformations exist whereby $z=f(x, y)$ can be considered

to be in the form

$$Z=X+Y+XY(X-Y)\sum_{0}^{k-3} q_{ij}X^iY^j+O(\varrho_{00}^k),$$

where $X=X(x)$, etc. Means for determining $X=X(x)$, etc., are not given. The coefficients of the Taylor expansions of the functions in Δ are obtained explicitly, in terms of the q_{ij} and arbitrary constants, for k up to 7, and necessary and sufficient conditions for determining them for $k=8$ and 9 are given. It follows that if $f(x, y)$ can be differentiated six times at x_0, y_0, and its first derivatives do not vanish there, $z=f(x, y)$ is nomographable to sixth order in the neighborhood of x_0, y_0. It is nomographable to the seventh order and even to the eighth "as a rule" since not to be so requires satisfying certain conditions among the q_{ij} (five in number for $k=7$). Ninth-order nomographability is attained only for functions which satisfy two conditions among the q_{ij}. Consideration is given to the vanishing of f_x or f_y or both. This did not appear in the authors' earlier brief report of this investigation [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 1137–1140; MR **16**, 633]. *R. Church.*

See also: Fan, p. 935; Roy and Sarhan, p. 936; Voronovskaia, p. 969; Birkhoff and Diaz, p. 1030.

Tables

★**Neville, E. H. Rectangular-polar conversion tables.** Royal Society Mathematical Tables, Vol. 2. Cambridge, at the University Press, 1956. xxxii+109 pp. $5.50.

This table gives, for $x=1(1)105$, $y=1(1)x$, values of $r=\sqrt{(x^2+y^2)}$ to 13D, of $\ln r$ to 15D and of $\theta=\arctan(y/x)$ in degrees to 13D and in radians to 15D. It is arranged as follows: on page n there is r, θ (in degrees) for $x=n$, $y=1(1)n$ and $\ln r$, θ (in radians) for $x=n(1)105$, $y=n$. In addition there are 15D values of $\ln x$, $x=1(1)160$ and $x \ln 10$, $x=1(1)20$. There is also a table of the (denominators of the) Farey series F_{105} of order 105 i.e. the sequence of rational numbers $f_i=y_i/x_i$ in ascending order and in their lowest terms where $0\leq y_i<x_i\leq 105$. There are various indices, a descriptive bibliography, and a detailed introduction, describing the construction and checking of the tables, and interpolation in them, illustrated by worked examples. This volume is more comprehensive and more professional than three recent ones by L. Ya. Nelšuler [Tables for the transformation of rectangular cartesian into polar coordinates, Gostehizdat, Moscow-Leningrad, 1950; MR **13**, 282], J. Todd [Tables of arctangents of rational numbers, Nat. Bur. Standards, Washington, D.C., 1951; MR **12**, 750] and Akademiya Nauk SSSR [Ten-place tables of logarithms of complex numbers and of the transformation from cartesian to polar coordinates, Izdat. Akad. Nauk SSSR, Moscow, 1952; MR **14**, 587].

The new material is mainly the values of θ. It was prepared essentially as follows. Since $x_iy_{i+1}-x_{i+1}y_i=1$ we have $\arctan f_{i+1}=\arctan f_i+\arctan n_i^{-1}$ where $n_i=x_ix_{i+1}+y_iy_{i+1}$ and so $\arctan f_i$ could have been built up from an initial value and the values of the differences, $\arctan n_i^{-1}$. Actually the sequence $\arctan n_i^{-1}$ was itself first built from its differences. More precisely, the sequence $\{n_i\}$ was arranged in order of magnitude and without repetitions as $\{m_i\}$ and then $\arctan m_i^{-1}$ was built up from its differences, $\arctan p_i$, where

$$p_i=(m_{i+1}-m_i)/(1+m_im_{i+1}).$$

In most cases $p_i<10^{-7}$ and so $\arctan p_i=p_i$ to about 20D. Many precautions were taken in the preparation: e.g., the sequences were built up from both ends, the building up was done for both the measures of the θ, instead of in one case, followed by a final multiplication to convert the tabular entry.

The values of r were obtained by new calculations and those of $\ln r$ were obtained from the National Bureau of Standards values [Tables of natural logarithms for arguments between zero and five to sixteen decimal places, Nat. Bur. Standards, Washington, D.C., 1953; MR **15**, 255], supplemented by special devices for determining uncertain end-figures.

Methods of interpolation which do not require the use of any additional tables are described. For instance, given x, y to find r, θ we first obtain consecutive entries a/b, c/d in F_{105} which bracket y/x. This is done by using the fact that the density of a Farey series is approximately uniform [see E. H. Neville, Proc. London Math. Soc. (2) **51** (1949), 132–144; MR **10**, 681]. Then r, θ can be obtained (each in two ways, for checking) as a power series in $u=(by-ax)/(bx+ay)$ or in $v=(cx-dy)/(cy+dx)$. Interpolation for $\ln r$ is done by means of a power series in $(r-q)/(r+q)$ or $(r^2-q^2)/(r^2+q^2)$, where q is a suitable tabular entry; some scaling is necessary if a small argument is wanted. Inverse interpolation is also discussed; this is needed, e.g., in the determination of the cartesian forms of powers of complex numbers. As an example a value of $(e+i\gamma)^\pi$ is computed.

Careful study of this volume, and in particular, the introduction, the format and printing is recommended to all table-makers: they will realize some of patient work of many collaborators which provided the solid basis for reputation of the group of table makers which was led by Neville for many years. *John Todd.*

★**Belen'kiĭ, N. S. Tablicy obratnyh čisel. [Tables of reciprocals.]** Gosudarstvennoe Statističeskoe Izdatel'stvo, Moscow, 1955. 311 pp. 16.35 rubles.

This main table gives n^{-1} to 7S for $n=10000(1)99999$. Each page contains six columns giving in all 300 reciprocals. It is comparable with the standard tables of W. H. Oakes [Table of the reciprocals of numbers, Layton, London, 1865] except that no differences are given by Belen'kiĭ. There is no indication of the sources of the table or the method of computation or checking. Spot checking using Oakes' table and that of National Bureau of Standards [Table of reciprocals of the integers from 100,000 to 200,009, Columbia Univ. Press, New York, 1943; MR **5**, 159] revealed no errors. *John Todd.*

★**Tables of the function** arc sin z. By the Staff of the Computation Laboratory. The Annals of the Computation Laboratory of Harvard University, vol. 40. Harvard University Press, Cambridge, Mass., 1956. xxxviii+586 pp. $12.50.

A branch of $\omega=f(z)=\arcsin z=u+iv$ maps the z-plane, cut along the real axis from $-\infty$ to -1 and 1 to ∞ on a strip $-\frac{1}{2}\pi\leq u\leq\frac{1}{2}\pi$ in the ω-plane. Because of symmetry the function is defined by its values in the first z-quadrant. The present volume gives 6D values of $f(z)$ for values of z in this region, together with appropriate means for interpolation. The latter are usually 6D values (of the real and imaginary parts) of the first and second

reduced derivatives (which are here called "pseudo-differences"). Near $z=1$, interpolation by means of auxiliary function is recommended. Let

$$(2-2z)^{-\frac{1}{2}}(\tfrac{1}{2}\pi-\arcsin z)$$

be expanded as a power series $g(1-z)$ in $1-z$. Then $g(t)$ and $g(-t)$ are tabulated, together with first differences, to 6D, for $t=0(.001).200$. From these we have $u=\frac{1}{2}\pi-(\frac{1}{2}t)^{\frac{1}{2}}g(t)$, $v=(\frac{1}{2}t)^{\frac{1}{2}}g(-t)$ where the argument t is a certain real function of x, y (different in the two cases).

Throughout, equal intervals in x and y are used and they increase from .002 in a region close to $z=1$ to 5 for values of z with $|z|$ between about 200 and 500. The interval of tabulation was chosen so that by use of the given derivatives, interpolation errors would not exceed 2×10^{-6}. Change of interval is made on a set of straight lines which bound a sequence of areas expanding from $z=1$. For larger values of z computation by an asymptotic series is recommended.

The table was prepared on the Harvard Mark IV Computer. The programming was done by J. Zahl; the Introduction was written by T. Singer, and includes worked examples of interpolation. For z small the function was computed directly and checked by computing the inverse function. For z larger, values were computed by a Taylor series with an imaginary increment, chosen so that at most 6 terms of the series would be needed. This meant that, on occasion, more values were computed than were needed to satisfy the interpolation-error criterion above; in this circumstance the redundant values were not printed. This part of the table was checked by computing a skeleton table directly.

It is clear that this table is also one for arcsinh z, arccosh z, arccos z. With it arctant z and arctanh z can also be obtained with a little care, since $\arctan z=\arcsin z(1+z^2)^{-\frac{1}{2}}=\frac{1}{2}\arcsin(2z/(1+z^2))$. This volume therefore makes all the inverse trigonometrical functions accessible to a reasonable precision in the complex domain; it should more than satisfy the needs of most users. In the earlier tables of R. Hawelka [Vierstellige Tafeln der Kreis- und Hyperbelfunktionen, sowie ihrer Umkehrfunktionen, im Komplexen, Vieweg, Braunschweig, 1931] the argument was in polar form and only four decimal values were given.

Reproduction is in the now familiar Harvard style, but the present page consists of 70 lines (instead of the usual 50) giving y, and the real and imaginary parts of f, $\delta f'$, $\frac{1}{2}\delta^2 f''$ for two values of x, and 70 of y. *John Todd.*

Huss, Carl R.; and Donegan, James J. Method and tables for determining the time response to a unit impulse from frequency-response data and for determining the Fourier transform of a function of time. NACA Tech. Note no. 3598 (1956), 38 pp.

The paper presents four-figure tables of the functions

$$f(z)=\frac{\sin z\cdot\cos(2n-1)z}{z} \text{ and } g(z)=\frac{\sin z\cdot\sin(2n-1)z}{z};$$

in both cases n runs through the integers from 1 to 50. The tabulation runs through $0\leq z\leq 2.50$ in steps of 0.01 for $f(z)$ and through $0\leq z\leq 1$ in steps of 0.05 for $g(z)$. The tables are to be applied for the numerical evaluation of integrals $h(t)=\int_0^\infty F(\omega)\cos\omega t\,d\omega$ such that the integral is written as $h(t)\approx\sum_{n=1}^{\infty}\Omega_n\cdot v_n$, where

$$\Omega_n=\int_{\omega_{n-1}}^{\omega_n}\cos\omega t\,d\omega,\quad v_n=(\omega_n-\omega_{n-1})^{-1}\int_{\omega_{n-1}}^{\omega_n}F(\omega)\,d\omega$$

and Ω_n is calculated by means of the tables. Some special examples and diagrams illustrate the use of the tables. The application is in the field of control systems, Fourier transforms and analysis of flight-test data. *H. Bückner.*

Slater, Lucy J. A short table of the Laguerre polynomials. Proc. Inst. Elec. Engrs. C. **103** (1956), 46–50.

This table gives

$$L_n(x)=\sum_{r=0}^{n}(-n)(-n+1)\cdots(-n+r-1)x^r/(r!)^2$$

to 5 or 6D for $n=0(1)10$, $x=0(.1)5$, together with second modified differences. It was computed by direct summation on EDSAC1 and checked by hand. There are some errors in the table, e.g., to 6D we have $L_8(4.3)=0.089391$ not 0.089400 as given. The author informs me that errors may have arisen when scaling difficulties caused a new routine to be used: this occurred for values of x exceeding 4.1, 3.4, 2.8, 2.4, 2.1, 1.9 in the cases $n=5, 6, 7, 8, 9, 10$ respectively. *John Todd* (Washington, D.C.).

★ **Miller, J. C. P., editor. Tables of Weber parabolic cylinder functions, giving solutions of the differential equation**

$$\frac{d^2y}{dx^2}+(\tfrac{1}{4}x^2-a)y=0.$$

Computed by Scientific Computing Service Limited. Mathematical introduction by J. C. P. Miller. Her Majesty's Stationery Office, London, 1955. (British Information Services, 30 Rockefeller Plaza, New York 20, N. Y.) 63s; $11.68.

A little more than one third of this volume is devoted to a mathematical introduction which is of considerable interest in itself; the remaining part contains numerical tables of certain functions which are related to Weber's differential equation but are not the functions usually called parabolic cylinder functions.

In the mathematical introduction, the two differential equations

(1) $d^2y/dx^2-(\frac{1}{4}x^2+a)y=0$ and (2) $d^2y/dx^2+(\frac{1}{4}x^2-a)y=0$

are considered side by side. Certain solutions, U, $\bar{U}$, V, $\bar{V}$ of (1) and $W(a,\pm x)$ of (2) are introduced; these solutions are real when a and x are real and have been chosen for their suitability for numerical tabulation. A self-contained and comprehensive study of these functions is given, including integral representations, convergent and asymptotic expansions, an excellent collection of formulas and a bibliography. [Since asymptotic expansions are discussed in great detail, and the bibliography is almost complete in this respect, references to the following could be usefully added: G. N. Watson, Proc. London Math. Soc. (2) **17** (1918), 116–148; S. C. van Veen, Akad. Wetensch. Amsterdam Proc. **34** (1931), 257–267; A. Kienast, Math. Z. **41** (1936), 739–753; A. Erdélyi, M. Kennedy, and J. L. McGregor, J. Rational Mech. Anal. **3** (1954), 459–485; MR **16**, 33.

The numerical part contains tables of $W(a,\pm x)$ and of certain related, slowly-varying, auxiliary functions. In the oscillatory region, the auxiliary functions F, G, χ, ψ are defined by

$$W(a,x)=k^{\frac{1}{2}}F\cos\chi,\qquad W(a,-x)=k^{-\frac{1}{2}}F\sin\chi,$$

$$\frac{dW(a,x)}{dx}=-k^{\frac{1}{2}}G\cos\psi,\quad\frac{dW(a,-x)}{dx}=-k^{\frac{1}{2}}G\sin\psi,$$

where $k=(1+e^{2\pi a})^{\frac{1}{2}}-e^{\pi a}$.

$W(a, \pm x)$ and reduced derivatives (for interpolation in the x direction) are tabulated for $a=-10(1)10$, $x=0(.1)10$; $\log W(a, \pm x)$ with first derivative for $a=1(1)10$, $x=0(.1)2a^{\frac{1}{2}}$; F, G, χ, ψ for $a=-10(1)2$, $x=0(.1)10$ and $a=3(1)10$, $x=2a^{\frac{1}{2}}(.1)10$; $W(a, 0)$ and $W'(a, 0)$ for $a=0(.02)1(.1)5(.2)20$; $e^{\pm\pi a}$, $k^{\pm\frac{1}{2}}$, $1-k^{\pm\frac{1}{2}}$ for $a=0(.1)10$; $e^{-\pi a}$ also for $a=0(.001).1$; and the real and imaginary parts of $\Gamma(\frac{1}{4}n+\frac{1}{4}ia)$ for $n=0(1)3$, $a=0(.1)5(.2)20$. An account of the preparation of the tables and instructions for interpolation are also given. *A. Erdélyi.*

Sharp, W. T.; Gove, H. E.; and Paul, E. B. Graphs of Coulomb functions. Atomic Energy of Canada, Ltd., Chalk River, Ont. Rep. **TPI-70** (1953, 2d ed. 1955), ix+58 pp. $1.50.

The authors set

$$A_L^2=F_L^2+G_L^2, \quad B_L^2=F_L'^2+G_L'^2,$$

$$\phi_L=\tan^{-1}\frac{F_L}{G_L}, \quad \psi_L=\sum_{m=1}^{L}\tan^{-1}\frac{\eta}{m},$$

where $F_L(\rho, \eta)$ and $G_L(\rho, \eta)$ are the Coulomb wave functions. They give graphs of A_L^2, B_L^2, ϕ_L, ψ_L for $L=0(1)4$, for various ranges of ρ, and $\log \eta=-.8(.1).6$, with a few supplementary graphs of quantities of importance in quantum mechanics. *A. Erdélyi.*

See also: Newman, p. 946; Borkmann und Oberländer, p. 975; Pillai, p. 983; Miller, p. 984; Nomachi, p. 984; Kuntzmann, p. 1009; Tyler and Rouleau, p. 1025.

Mathematical Machines

Badillo Barallat, M. C. Representation schemes for systems controlled by a multivalued logic. Calc. Automat. Cibernet. **4** (1955), no. 9, 54–62. (Spanish. English summary)

L'auteur considère des circuits comportant, au lieu des interrupteurs ordinaires, ce qu'il appelle des „roues dentées électroniques". Une roue denteé électronique „a n dents", autrement dit, susceptible de se trouver dans n états différents, et qui se trouve dans l'état k, $0\leq k\leq n-1$, ne laisse passer que les trains constitués par au moins $n-k$ impulsions-types. $n-k$ est appelé l'indice de l'état k. Deux roues d'indices r et s sont branchées en série (respectivement en parallèle) lorsque le branchement ne laisse passer que des trains constitués par min (r, s) (resp. max (r, s)) impulsions-types au plus.

L'algèbre qui correspond aux branchement série-parallèle de telles roues est précisément celle d'une logique multivalente. Le texte est par endroits fort obscur. [Dans la référence à Shannon lire „57–1938" au lieu de „95–1948".] *J. Riguet* (Paris).

Perkins, Robert. EASIAC, a pseudocomputer. J. Assoc. Comput. Mach. **3** (1956), 65–72.

Stiefel, E. Elektronische Rechenautomaten als Beispiel zur allgemeinen Kybernetik. Actes Soc. Helv. Sci. Nat. **135** (1955), 53–62.

Boehm, Carl; and Röper, Gerhard. Elektronische Rechenmaschinen und Informationsverarbeitung. Bl. Deutsch. Ges. Versicherungsmath. **2** (1956), 485–510.

Hunt, P. M. The electronic digital computer in aircraft structural analysis. The programming of the Argyris matrix formulation of structural theory for an electronic digital computer. III. General purpose programmes for the force and displacement methods in large structures including the use of magnetic tape storage. Aircraft Engrg. **28** (1956), 155–165.

Blecher, Franklin H. Transistor circuits for analog and digital systems. Bell System Tech. J. **35** (1956), 295–332.

Mitrović, Dušan. Sur certaines applications des organes à fonctionnement discontinu dans le calcul analogique. Bull. Inst. Nuclear Sci. "Boris Kidrich" no. **50** (1954), 1–11.

Aleksić, Tihomir Ž. Amplitude comparator for computing purposes. Bull. Inst. Nuclear Sci. "Boris Kidrich" no. **51** (1954), 13–19.

Aleksić, Tihomir Ž. Simple electronic extrapolations of sampled data. Bull. Inst. Nuclear Sci. "Boris Kidrich" no. **76** (1955), 37–42.

Spearman, F. R. J.; Gait, J. J.; Hemingway, A. V.; and Hynes, R. W. Tridac, a large analogue computing machine. Proc. Inst. Elec. Engrs. B. **103** (1956), 375–390; discussion 390–395.

Mitrović, Dušan; and Tomović, Rajko. An analogue computer for solving linear simultaneous equations. Bull. Inst. Nuclear Sci. "Boris Kidrich" no. **17** (1953), 5–11.

Hayashi, Yoshio. On a machine solving algebraic equations of higher order with real coefficients. Sci. Papers Fac. Engrg. Tokushima Univ. **2** (1950), no. 1, 61–63. (Japanese. English summary)

Bodenstedt, Erwin. Analog computer for the differential equation $y''+f(x)y+g(x)=0$. Rev. Sci. Instrum. **27** (1956), 218–221.

Položiĭ, G. N. Effective solution of a problem on approximate conformal mapping of simply connected and doubly connected regions and determination of the Christoffel-Schwarz constants by means of electrohydrodynamical analogies. Ukrain. Mat. Ž. **7** (1955), 423–432. (Russian)

Šamanskiĭ, V. E. On conformal mapping by means of an electrical analogy. Ukrain. Mat. Ž. **8** (1956), 92–96. (Russian)

RELATIVITY

★**Synge, J. L. Relativity: the special theory.** North-Holland Publishing Co., Amsterdam; Interscience Publishers, Inc., New York; 1956. xv+450 pp. $10.50.

A full-scale treatment of the special theory of relativity, with particular emphasis on the expression and interpretation of its concepts and results in terms of the Minkowski geometry of flat space-time. The approach is in the main deductive; the mathematical structure is erected a priori, and its theorems fleetingly interpreted in terms of physical experience. This reversal of the historical role of experiment enhances the logical clarity of the

exposition, but at some sacrifice of exciting episodes in the history of thought — as reflected for example in the work of Fresnel, Airy, Michelson and Einstein himself. Familiarity with elementary vector and tensor analysis is assumed, but special topics of particular importance are developed as required, either in the text or in appendices at the end of the book.

Chapters I–III are devoted to the geometrical structure of the 4-dimensional world of events, with its fundamental distinction between time-like, null and space-like intervals and their measurement. In keeping with his formal approach the author elects to employ the imaginary Minkowski coordinate $x_4=ict$ in place of the real time t, and at times makes use of the purely extraneous Euclidean metric imposed by the plane on which his schematic diagrams are drawn. These are followed by Chapter IV on Lorentz transformations, with an illuminating account of the role played by invariant null lines and of the relation to spinors. The final chapter of the five dealing with the geometrical or kinematical aspects, constituting together over one-third of the book, applies the Lorentz transformations to problems of motion: FizGerald contraction, Doppler effect, aberration, the Michelson-Morley experiment, and a brief account of Milne's special relativistic model of the expanding universe of nebulae.

Chapters VI and VII deal with the mechanic of particles and with collision phenomena involving particles, photons and "internal impulses" — the latter introduced as a means of transmitting energy and momentum from one particle to another at a distance. Here the leading rôle is played by the conservation laws, including that of orbital and intrinsic angular momentum. The difficult collision problem for two particles of different masses and spins is solved, and summarized with a practical procedure for carrying it out. The 2-body problem is treated by considering the interaction as supplied by internal impulses, resulting in a force which is in the classical approximation the same in form as Newtonian gravitation. The final Chapter VIII of this second third of the book develops the mechanics of a continuum, in terms of the energy-momentum-stress tensor T_{rs}, and explores the relation of the continuum to the discrete representation.

The final Chapters IX and X consider electromagnetic fields and their production by moving charges. The electromagnetic field tensor F_{rs} and the resulting T_{rs} are given elegant geometrical interpretations which throw much light on the invariantive structure of fields. Electromagnetic models of a material particle and of a photon — the latter reminiscent of H. Bateman's moving singularities — are built up from elementary potentials. Finally, the ponderomotive force and the equations of motion of a charged particle are set up, and the latter integrated for the case of Kepler motion. Finally, the general form and theory of Maxwell's equations are given, and the 4-potential introduced.

In brief, Synge's book is a quite complete account of the mathematical structure and content of the special theory of relativity. The author promises that it will be followed by a companion volume on the general theory.

H. P. Robertson (Paris).

Gürsey, Feza. Contribution to the quaternion formalism in special relativity. Rev. Fac. Sci. Univ. Istanbul. Sér. A. 20 (1955), 149–171 (1956). (Turkish summary)

The author develops a formalism involving complex quaternions and applies this to the study of the Lorentz group in Minkowski space-time. The results obtained are already known from the theory of two-component spinors as developed by Veblen and others. *A. H. Taub.*

Matsumoto, Toshizô. Sur la déduction axiomatique des formules de transformation de Lorentz. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 29 (1955), 55–58.

A quasi-geometrical derivation of the Lorentz transformations, based on the kinematical principle of relativity. The postulate of the constancy of the velocity of light is replaced by assumptions concerning the contraction factors and the equality of angles between certain of the coordinate axes, the physical meaning of which is rather obscure. *H. P. Robertson* (Pasadena, Calif.).

★ **Arzeliès, Henri. La cinématique relativiste.** Gauthier-Villars, Paris, 1955. xi+229 pp. 2500 francs.

This is an exhaustive study of kinematic relativity containing a very rich quotation of literature.

L. Infeld (Warsaw).

★ **Jordan, Pascual. Schwerkraft und Weltall. Grundlagen der theoretischen Kosmologie. 2te Aufl.** Friedr. Vieweg & Sohn, Braunschweig, 1955. xi+277 pp. DM 16.80.

The first three chapters, well over half the book, give an elegant account of Einstein's general theory of relativity and its offshoot, projective relativity. The changes from the first edition [1952; MR 14, 1022] are mainly additions to provide further background for Jordan's own extension, to which the remaining two chapters are devoted. These include an account of various forms of the de Sitter line element (Appendix to Chapter II), reasons for considering a closed cosmological model (in Section 20), and the relationship of the Kaluza-Klein 5-dimensional relativity to the projective formulation (Appendix to Chapter III).

The author's extended theory, with a variable gravitational "constant", is treated in a drastically revised and extended Chapter IV. The formulation of the field equations and the account of Heckmann's spherically symmetric solution are pretty much as before, but a powerful new tool suggested by Pauli — the formal invariance of the field equations on going over to a conformal metric — has been added and used to advantage. The mathematical theory of the cosmological model (which in the original edition wound up in a revolting differential equation) has been pushed, with the assistance of E. Schücking and H. König, to a point where its general characteristics are much better understood. Further work on the non-static spherically symmetric field is interpreted as giving support to the possibility of sudden appearance in our world of mass which had previously pursued an independent existence — a possibility exploited later in Jordan's hypothesis on the birth of stars. Book 2 ends with a speculative Section 34, on possible implications of variable gravity for other scientific fields, from geology to the cosmogony of double stars.

The last Book 3, "The hypothesis of the generation of matter", consists in a new Chapter V, "Embryonic stars", which drastically revises and extends Jordan's previous speculations in this field. Here even the criterion laid down by the author in his preface — that the reviewer has at least looked at (or even read) the book — is no guarantee of the adequacy of a review, but the following brief words may give some idea of the content of this

chapter. Jordan recounts some of the difficulties astronomers are now having in accounting for the evolution of stars and stellar systems, and certain of the paths that are being probed in the effort to overcome them —Ambarzumian's stellar associations, Baade's two populations, hypotheses on planetary nebulae, novae and supernovae, and the like. Still the author finds it unlikely that order can be brought into the world-system without some radical new principle, and this he believes he has in the hypothesis of the birth of stars sketched in his last Section 38. The kernel of his theory lies in the exploding of matter into our spatial frame, for the possibility of which he finds support in the non-static solutions to which he has been led by his theory of the variable gravitational field. *H. P. Robertson.*

Just, Kurt. Jordansche Kosmologie mit neuen Feldgleichungen. Z. Physik **141** (1955), 592–603.

The author integrates one form of the field equations describing a gravitational field with a varying gravitational "constant" under the following assumptions: (a) the space-time is a cosmological space, (b) the stress-energy tensor is that of a perfect fluid with negligible pressure, and (c) a quantity b which measures the variation of the Lagrangian function with respect to the varying gravitational "constant" is negligible. Various speculative arguments are put forward to evaluate various constants of integration and interpret the model of the universe obtained. *A. H. Taub* (Urbana, Ill.).

Just, Kurt. Superpotentiale in der erweiterten Gravitationstheorie. Z. Physik **142** (1955), 493–502.

The author examines the analogues of the Bianchi identities for Jordan's generalisation of the theory of gravitation in which the gravitational "constant" is variable. The identities in question are obtained by making use of the fact that the Lagrangian used is invariant under coordinate transformations. Einstein's theory of general relativity is also discussed and it is shown that the pseudo-tensor describing the energy and momentum of the gravitational field is not unique. The corresponding pseudo-tensor for the Jordan theory is given and said to have no physical significance. *A. H. Taub.*

Ludwig, G.; and Just, K. Zur Jordanschen Theorie der Materie-Entstehung. Z. Physik **143** (1955), 472–476.

The authors discuss various possible values of constants which enter into the Lagrangian which is to be used for the derivation of the field equations of the Jordan theory of relativity in which the gravitational "constant" varies. *A. H. Taub* (Urbana, Ill.).

Just, Kurt. Die Drehwaage von Eötvös bei veränderlicher Gravitationszahl. Z. Physik **144** (1956), 648–655.

The author discusses the classical experiments of Eötvös from the view point of the motion of a test body in a central field in the Jordan theory of relativity in which the gravitational "constant" varies and in which gravitational and inertial mass are not identical. The implication of these experiments concerning the constants entering into the Lagrangian from which the field equations are derived are said to be indecisive. *A. H. Taub* (Urbana, Ill.).

★**Schrödinger, E. Expanding universes.** Cambridge, at the University Press, 1956. viii+93 pp.

In this short book the author presents a somewhat novel discussion of mainly known results regarding various topics in cosmological spaces. The first chapter is devoted to the geometry of De Sitter space. The various representations of this space as (a) a 4-dimensional subspace of constant curvature in a 5-dimensional flat space, (b) a static 4-dimensional space-time and (c) a space-time with the Lemaître-Robertson line element

$$ds^2=-e^{-2t}d\sigma^2+R^2dt^2,$$

are discussed and compared.

Chapter II discusses the non-null and null geodesics in a general space-time and applies these results to discuss the Doppler effect in cosmological spaces. Chapter III points out the inadequacy of the geometrical optics used in Chapter II and outlines an approximate wave optics treatment of the behavior of light and material waves in the presence of gravitational fields. The Hamilton-Jacobi theory of special relativity is extended to general relativity. Chapter IV discusses the solution of the scalar wave equation in the space-time of an expanding universe. *A. H. Taub* (Urbana, Ill.).

Bochner, S. Stationary space-time in general relativity. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 485–490.

L'étude, en relativité générale, des espaces-temps V_4 stationnaires, extérieurs partout conduit à l'étude, sur une variété V_3 de métrique définie positive, de la relation [Lichnerowicz, Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955; MR 17, 199]

$$\Delta\xi=-\xi R+\tfrac{1}{4}\xi^3H^2$$

où ξ, R et H^2 sont des scalaires ($\xi>0$, $H^2\geqq 0$) et où Δ est l'opérateur laplacien sur V_3. Pour des raisons physiques évidentes, l'A. nomme „vide" un V_4 pour lequel $H=0$, $\xi=$const et „semi-vide" un V_4 pour lequel $H=0$. Le but du travail est de donner des conditions sous lesquelles on puisse affirmer que V_4 est vide ou semi-vide. Après avoir rappelé les résultats classiques relatifs au cas où V_3 est supposé compact, l'A. n'assume plus la compacité de V_3, mais se place dans l'hypothèse suivante: il existe sur V_3 un ensemble compact S et un groupe Γ de transformations V_3 tels qu'à tout point x de V_3, on puisse associer un élément γ de Γ pour lequel $\gamma x \in S$.

Un tenseur t de V_3 est dit presque périodique relativement à Γ si toute suite infinie d'éléments de Γ admet une suite partielle $\{\gamma_n\}$ jouissant de la propriété suivante: tout point de V_3 admet un voisinage dans lequel la suite $t(\gamma_n x)$ converge uniformément et si $t^*(x)$ est la limite, la suite $t^*(\gamma_n{}^{-1}x)$ converge uniformément vers $t(x)$. Le principal résultat du papier est le suivant: sous l'hypothèse faite sur V_3, si le tenseur metrique, ξ est ses dérivées des deux premiers ordres, R et H^2 sont presque périodiques relativement à Γ et si $R\leqq 0$, V_4 est vide. L'A. envisage aussi le cas particulier où, V_3 étant homéomorphe à R^3 et Γ étant le groupe des translations, la presque périodicité au sens de Bohr peut être introduite pour les composantes. Dans ce cas la substitution à $R\leqq 0$ de l'hypothèse plus faible $M(\xi R)\leqq 0$ (où M est l'opérateur de moyenne) permet encore d'affirmer que V_4 est semi-vide. Une extension de ce résultat au cas général de la presque périodicité relativement à un groupe est indiquée. *A. Lichnerowicz* (Paris).

Schmutzer, Ernst. Bemerkungen zum einheitlichen Variationsprinzip für die mechanischen, elektrodynamischen und Gravitations-Grundgesetze. Z. Physik **143** (1955), 479–488.

In the first part of this paper the author discusses the

difficulties occurring in classical theory when one tries to derive the Maxwell equations, the Einstein gravitational equations and the Lorentz pondermotive force equations from a single variational principle. In the second part of the paper the author represents matter in terms of a scalar wave function and obtains a single variational principle which the Maxwell equations, the Einstein gravitational equations and the Klein-Gordan equation may be derived. *A. H. Taub* (Urbana, Ill.).

Just, Kurt. Strenge kugelsymmetrische Lösungen der Einsteinschen Feldgleichung. Z. Physik **145** (1956), 235–240.

The author discusses the solution of the Einstein field equations for a metric assumed to be of the form

$$ds^2=e^{\lambda}dr^2+e^{\nu}(d\theta^2+\sin^2\theta d\varphi^2)-dt^2.$$

This problem has previously been discussed in greater detail by H. Bondi [Monthly Not. Roy. Astr. Soc. **107** (1947), 410–425; MR **10**, 214]. *A. H. Taub.*

Penfield, R. H.; and Zatzkis, Henry. On the determination of the equations of motion from a general, covariant, non-linear field theory by the approximation method of Einstein, Infeld, and Hoffman. Acta Phys. Austriaca **10** (1956), 87–94.

The authors develop some ideas of P. Bergmann [Phys. Rev. (2) **75** (1949), 680–685; MR **10**, 408] in which Lagrangian density of the form $\mathcal{L}=\Lambda^{A\varrho B\sigma}y_{A,\varrho}y_{B,\sigma}$ is used (y_A and $\Lambda^{A\varrho B\sigma}$ denote respectively field variables and some coefficients, which are functions of y_A). Generalized Bianchi identities $F^{B\nu}{}_{A\mu}(y_BL^A)_{,\nu}+y_{A,\mu}L^A\equiv 0$ are deduced from general covariance of the theory. Here $F_{A\mu}{}^{B\nu}$ are numbers which depend on the type of field variables and

$$L^A\overset{\text{def}}{\equiv}\frac{\partial\mathcal{L}}{\partial y_A}-\left(\frac{\partial\mathcal{L}}{\partial y_{A,\varrho}}\right)_{,\varrho}.$$

Using these identities and the "new approximation" method it is shown that field equations determine the motion of singularities in an arbitrary approximation order. *I. Birula-Białynicki* (Warsaw).

Géhéniau, J.; et Debever, R. Les invariants de courbure de l'espace de Riemann à quatre dimensions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 114–123.

The authors obtain expressions for the independent curvature-invariants, fourteen in number, of 4-dimensional Riemannian space. By definition these are scalar functions of the metric tensor g_{kl} and of its first and second derivatives. The invariants are built up out of the metric tensor, the alternating tensor ε_{ijkl} and the curvature tensor. They assume specially simple forms when referred to a Ricci principal frame. *H. S. Ruse* (Leeds).

Gomes, Ruy Luis. General relativity. The curvature tensor of an absolute space. Univ. Lisboa. Rev. Fac. Ci. A. (2) **4** (1955), 245–262. (Portuguese)

The author computes the curvature tensor for a linear element in which the time lines are orthogonal trajectories of the spaces x^4=const. He considers in particular the cases in which x^4 is an "absolute time" (i.e., g_{44} independent of the spatial coordinates x^α) and in which the spaces are "rigid" (i.e., $g_{\alpha\beta}$ independent of x^4), and discusses the general conditions that the spatial components of the space-time curvature tensor coincide with those of the curvature tensor of the 3-spaces x^4=const. *H. P. Robertson* (Pasadena, Calif.).

Weiszmann, A. Du caractère physique des espaces métriques relativistes. Acad. R. P. Romîne. Fil. Iași. Stud. Cerc. Ști. **6** (1955), 201–208. (Romanian. Russian and French summaries)

Vescan, T. T.; Mihul, E.; et Ioniță, G. Recherches sur la théorie de la relativité generalisée et la théorie des corpuscules spatialement étendues. Acad. R. P. Romîne. Fil. Iași. Stud. Cerc. Ști. **6** (1955), 217–226. (Romanian. Russian and French summaries)

Eisenhart, Luther P. A unified theory of general relativity of gravitation and electromagnetism. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 249–251.

The Einstein unified connection based on a general tensor $g_{\lambda\mu}=h_{\lambda\mu}+k_{\lambda\mu}$ ($h_{\lambda\mu}\overset{\text{def}}{\equiv}g_{(\lambda\mu)}$, $k_{\lambda\mu}\overset{\text{def}}{\equiv}g_{[\lambda\mu]}$) is

$$\Gamma_{\lambda\mu}{}^{\nu}=\left\{\begin{matrix}\nu\\ \lambda\ \mu\end{matrix}\right\}+S_{\lambda\mu}{}^{\nu}+U^{\nu}{}_{\lambda\mu},\quad S_{\lambda\alpha}{}^{\alpha}=0, \tag{1}$$

where

$$S_{\lambda\mu}{}^{\nu}\overset{\text{def}}{\equiv}\Gamma^{\nu}_{[\lambda\mu]},\quad U^{\nu}{}_{\lambda\mu}=2k^{\nu\alpha}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}$$

[Hlavatý, J. Rational Mech. Anal. **2** (1953), 1–52; MR **14**, 595]. The author's connection is a special case of (1) for

$$U^{\nu}{}_{\lambda\mu}=0,\quad S_{[\lambda\mu\nu]}=0 \tag{2}$$

[Hlavatý, ibid. **3** (1954), 147–179; MR **15**, 654] with the field equations

$$R_{\mu\lambda}=0, \tag{3}$$

$R_{\mu\lambda}$ being the contracted curvature tensor of $\Gamma^{\nu}_{\lambda\mu}$. [Remark of the reviewer: The second conditions (1)(2) show that we are dealing with a pure radiation. The conditions (1)(2)(3) are satisfied for the plane wave of the electromagnetic theory of light.] In the second part the author shows that

$$h_{\lambda\mu}v^{\lambda}v^{\mu}=\text{const}\ \left(v^{\lambda}\overset{\text{def}}{\equiv}\frac{dx^{\lambda}}{ds}\right)$$

is compatible with

$$v^{\nu}\nabla_{\nu}v^{\lambda}-a_{\nu\lambda}v^{\nu}=0, \tag{4}$$

where ∇_{ν} is the symbolic vector of covariant derivative with respect to $h_{\lambda\mu}$ and $a_{\nu\lambda}=a_{[\nu\lambda]}$ is an arbitrary tensor. Substituting into (4) the electromagnetic tensor multiplied by e/m one obtains the classical relativistic Lorentz formula. *V. Hlavatý* (Bloomington, Ind.).

Shaw, R. Spinor identities. Proc. Cambridge Philos. Soc. **51** (1955), 234–236.

Four elements γ_μ, with the usual rules $\gamma_\mu\gamma_\nu=-\gamma_\nu\gamma_\mu$ for $\mu\neq\nu$ and $-(\gamma_0)^2=(\gamma_1)^2=(\gamma_2)^2=(\gamma_3)^2=1$, generate a group $\{\pm\gamma^A\}$ of order 32. The orthogonality relations in a matrix representation of this group are transformed in various ways, using the covariant matrices $B\gamma_\mu=(B\gamma_\mu)^T$, where $B^T=-B$, into Lorentz invariant forms. Multiplying by four anticommuting spinor fields yields the Wigner-Chritchfield interaction in terms of invariants of β-decay for a general representation of the γ-matrices. *W. Givens* (Knoxville, Tenn.).

Brown, G. Burnlston. A theory of action-at-a-distance. Proc. Phys. Soc. Sect. B. **68** (1955), 672–678.

A phenomenological theory of the force between particles in relative motion, with discussion of its physical consequences. The author takes as the lowest (second) order velocity-dependent terms those given by Ampères'

law for the force between two current elements, augmented by terms arising from a potential, the contribution of which vanish on integrating around a closed circuit. The coefficient of the potential is evaluated by requiring that the perihelion advance of a planet be the same as in the general theory of relativity. The dependence on acceleration is taken over from the theory of the dipole oscillator, and on application of Mach's hypothesis concerning the origin of inertia leads to the relation $G=c^2R/M$ for the gravitational constant, where R and M are the radius and total mass of the "universe". The author claims to derive, in second order, the usual expression for the transverse mass of a particle (but the reviewer is unable to check his analysis). *H. P. Robertson* (Pasadena, Calif.).

Popovici, A. Les bases experimentelles et théoriques de la théorie des constantes physiques. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 1 (1952), no. 1, 77–98. (Romanian. Russian and French summaries)

From the author's summary: „Nous venons préciser les bases experimentelles et théoriques d'une théorie unitaire des constantes physiques universelles (continues et quantiques) liée à la théorie des groupes finis" [development of the same author's paper in Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 3 (1951), 417–427; MR 15, 585]. *F. A. E. Pirani* (London).

Popovici, Andrei. Déduction variationnelle des équations gravifiques et électromagnétiques, conformes covariantes de II^e ordre. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6 (1954), 65–99. (Romanian. Russian and French summaries)

L'A. indique comment on peut déduire les équations gravitationnelles et électromagnétiques de sa théorie [Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 2 (1953), no. 3, 78–131; MR 17, 908] d'un principe variationnel basé sur l'hamiltonien $H=(\frac{1}{2}\Gamma^2+kF^2)\sqrt{g}$, où Γ est une courbure scalaire conforme et F un tenseur antisymétrique correspondant au champ électromagnétique. Les équations obtenues présentent l'invariance conforme. Les relations de conservation correspondantes sont explicitées et le cas du champ statique à symétrie spherique étudié. Quelques remarques concernant la théorie du champ non symétrique, l'électrodynamique non linéaire et les aspects ondulatoires terminent le papier.

A. Lichnerowicz (Paris).

Verschaffelt, J. E. Sur l'écoulement relatif d'un fluide. J. Phys. Radium (8) 17 (1956), 313–319.

See also: Ghosh, p. 935; Pignedoli, p. 1018; Marx, p. 1029.

MECHANICS

★**Beghin, Henri. Cours de mécanique théorique et appliquée, à l'usage des ingénieurs et des étudiants de facultés professé à l'Ecole Polytechnique. Tome I.** Gauthier-Villars, Paris, 1952. x+551 pp. 900 francs.

Il y a en langue française plusieurs cours de mécanique de même origine, mais ce livre-ci exprime déjà par son titre quelque part une programme ou au moins une tendance nouvelle. Non seulement l'auteur donne dans le texte et dès le commencement beaucoup de remarques qui sont intéressantes pour un futur ingénieur, mais dans l'ouvrage entier on rencontre le sens du concret, „cette démarche de l'esprit qui ne sépare pas une propriété de l'objet qu'elle caractérise" (selon la préface du général Brisac), un éloignement de tout dogmatisme et l'interprétation que la mécanique est une science expérimentale. On ne s'étonne point que dans l'introduction l'auteur a cité Mach et son principe de la moindre dépense intellectuelle. Ne transiguent nulle part avec l'exactitude mathématique il a donné par ses idées à son livre une concision et une vivacité remarquables. Le chapitre I donne la cinématique du point et du solide avec les notions sur les engrenages et les systèmes articulés; II la géométrie des masses, III, IV et V les lois fondamentales de la dynamique, VI les actions de contact. Le chapitre VII, intitulé „Exercises simples de mécanique terrestre" est attrayant par un grand nombre d'exemples intéressantes et quelquefois anecdotiques (saut en profondeur, embarcation sur houle, cage contenant un oiseau, mouvement sur un plateau mobile). IX traite travail et puissance, X les systèmes d'unités, XI la mécanique du point matériel avec beaucoup d'exercises, XII les théorèmes du travail et de la force vive, XIII la mécanique des fils, XIV les équations de Lagrange, XV chocs et percussions, XVI les oscillations et les mouvements voisins d'un mouvement stationnaire, XVII et XVIII les équations d'Appell et quelques notions sur la mécanique analytique.

O. Bottema (Delft).

★**Beghin, Henri. Cours de mécanique théorique et appliquée, à l'usage des ingénieurs et des étudiants de facultés. Tome II.** Gauthier-Villars, Paris, 1951. 328 pp.

Ce second volume contient les parties plus élevées de la mécanique du solide et des milieux continus, toujours avec un interêt vif pour les applications techniques. Les chapitres I, II traitent le mouvement du corps solide avec de nombreuses exemples (toupie, gyroscope, gyrocompas, stabilisation), III, IV les machines (energie, régularisation du mouvement), V, VI les mouvements d'un corps déformable, VII l'équilibre et VIII les mouvements des fluides parfaits, IX et X les mouvements irrotationels d'un fluide incompressible jusqu'au théorèmes de Blasius et de Kutta-Joukowsky, XI la théorie tourbillonnaire, XII la propagation des ondes, XIII les notions de viscosité et les mouvements turbulents. Les chapitres XIV et XV sont dévoués à l'équilibre des solides élastiques et à la résistance des matériaux; XVI traite les théorèmes de travail (Clapeyron, Castigliano, Maxwell-Betti), XVII les petits mouvements des corps élastiques et le chapitre XVIII sur la similitude mécanique termine cet ouvrage excellent et plein de richesse. *O. Bottema* (Delft).

★**Hertz, Heinrich. The principles of mechanics.** Preface by H. von Helmholtz. Translation by D. E. Jones and J. T. Walley. Introduction by Robert S. Cohen. Dover Publications, Inc., New York, 1956. xlii+274 pp. Paperbound: $1.75; clothbound: $3.50.

Except for the new Introduction by R. S. Cohen this is a photo-offset reproduction of the edition of 1899 [Macmillan, London and New York].

Kukles, I. S. On some questions of the methodology of mechanics. Uzbek. Gos. Univ. Trudy (N.S.) no. 58 (1955), 3–53. (Russian)

Lilić, Borislav. Contribution à une conception plus précise des forces d'inertie. Bull. Soc. Math. Phys. Serbie **6** (1954), 209–233. (Serbo-Croatian. French summary)

Litvin-Sedoĭ, M. Z. On the kinematics of the free motion of a right trihedron. Vestnik Moskov. Univ. **10** (1955), no. 10, 59–64. (Russian)

[It would be impractical to present the following personal impressions, and to use the notation of the paper.] Let B/A denote the orthogonal 3-by-3 matrix of the cosines of the angles between the axes of a right trihedron B and those of a congruent trihedron A of the same sense and the same origin 0. Let N_i ($i=1, \cdots, n$) be axes through 0, the angle between N_i and N_{i+1} being λ_i. Let Q_i be an axis through 0 perpendicular to N_i and N_{i+1}, the angle between Q_i and Q_{i+1} being ϕ_i. Let A_i be the right trihedron $N_iP_iQ_i$. Then the basic result of the paper is:

$$A_n/A_1=\sum_i \Phi_i\Lambda_i \quad (i=1, \cdots, n-1),$$

where Φ_i and Λ_i have, respectively, the rows:

$$1, 0, 0;\ 0, \cos\phi_i, \sin\phi_i;\ 0, -\sin\phi_i, \cos\phi_i;$$

$$\cos\lambda_i, \sin\lambda_i, 0;\ -\sin\lambda_i, \cos\lambda_i, 0;\ 0, 0, 1,$$

[The prolixity of notation, the abundance of misprints, and various pranks of the printer's devil have discouraged the reviewer from delving deeper into the paper.]

A. W. Wundheiler (Chicago, Ill.).

Vâlcovici, V. Sur les équations générales du mouvement d'un solide à liaisons. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 77–91. (Romanian. Russian and French summaries)

L'auteur étudie sept cas, dont nous nommons les suivants: un des points du solide a un mouvement donné; un point est obligé de rester sur une courbe donnée et un autre sur une surface donnée; quatre points sont obligés de rester sur une surface donnée. *O. Bottema.*

Kuhtenko, A. I. On a class of mechanisms with nonholonomic constraints. Trudy Inst. Mašinoved. **15** (1955), no. 58, 46–71. (Russian)

In connection with some regulating mechanisms and mechanical integrators, Appell's equations are written down for systems with two degrees of freedom constrained by one equation linear and homogeneous in the velocities. Most of the material is expository. The main purpose of the paper is, presumably, to bring the theory of nonholonomic constraints within reach of the engineer not trained in analytical dynamics.

A. W. Wundheiler (Chicago, Ill.).

Hanus, W. Hamiltonian formalism and canonical commutation relations in the case of first order Lagrange equations. Acta Phys. Polon. **14** (1955), 309–321. (Russian summary)

The paper deals with dynamical systems which are described by first-order Lagrange equations. For such systems the transition from the Lagrangian formulation to the Hamiltonian formulation can be accomplished only if the commutator of the generalized coordinates and the generalized momenta differ from the customary form by a factor of $\frac{1}{2}$. The usual methods of obtaining this form, such as the variational methods, can still be applied if one takes into account that the generalized momenta are now linear combinations of the generalized coordinates. The present paper, however, uses the method based on the correspondence between the classical Poisson brackets and quantum-mechanical commutators. To this end the author considers a classical system with a finite number of degrees of freedom and assumes a Lagrangian of the form

$$L=A_{ik}\dot{q}_j''q_k'-A_{kj}\dot{q}_j'q_k''-K_{jk}q_k'q_j'',$$

where q_j' and q_j'' ($j=1, \cdots, f$) are two groups of generalized coordinates characterizing the system with $2f$ degrees of freedom. It is then shown that one can consider the $4f$ variables q_j', q_j'', p_j', p_j'' as canonical variables if the Hamiltonian is "symmetrized" by a transformation given in the paper. The relationship between the q_j's and p_j's is then a canonical one independent of time. The quantization can also be carried out then by the usual correspondence method, although the q's and p's are not linearly independent, and neither are the commutation relations resulting from the quantization. Both the classical Poisson brackets and the commutation relations contain the factor $\frac{1}{2}$ on the right-hand side. It is shown that the above procedure can be used only for one of the infinitely many Lagrangians of the system (differing in the added total time derivative). This one Lagrangian is symmetric with respect to integration by parts, and is therefore distinguished among the infinite set of Lagrangians. In two further special cases although the above procedure cannot be used, half the variables can be eliminated immediately in which case the usual situation of linear independence prevails. Finally it is shown that nothing in the above considerations depends on the finiteness of the number of degrees of freedom and hence the results apply to field like the Dirac and the Kemmer fields. *M. J. Moravcsik* (Upton, N.Y.).

Četaev, N. G. On the equations of motion of a body of invariable shape. Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 5–7. (Russian)

The paper briefly indicates an "analytic proof" of Zeeliger's equations [Theory of the motion of a body of invariable shape, Kazan, 1892] based, presumably, on the Boltzmann-Woronetz-Hamel's equations in quasicoordinates [E. T. Whittaker, Analytical dynamics, 4th ed., Cambridge, 1937, p. 43]. Zeeliger's equations consist of the three equations for the mass center, three equations of angular momentum (Euler), and the equation (in slightly changed notation),

$$d\Pi\eta/dt=Ap^2+Bq^2+Cr^2+\Pi\eta^2+\Sigma_i\,(OM_i)\cdot F_i,$$

where $\Pi=A+B+C$, η the rate of dilatation, F_i the force applied at the point M_i, and O the mass center. (The last term is the virial.) The paper uses Lie operators, assumes the existence of a potential, and does not mention Liouville's 1858 equations [J. Math. Pures Appl. (2) **3** (1858), 1–25] for the general case of a system with seven degrees of freedom, nor any other pertinent work prior to 1892 [see E. J. Routh, Advanced rigid dynamics, 6th ed., Macmillan, London, 1905, arts. 22–24].

A. W. Wundheiler.

Pignedoli, Antonio. Sulla dinamica delle particelle di energia relativistica: moto relativo di due elettroni veloci, i quali si attraggano mutuamente secondo la legge elettrodinamica di Weber. Boll. Un. Mat. Ital. (3) **11** (1956), 10–15.

The author studies the relative motion of two electrons which attract (sic) one another according to Weber's law,

and whose masses vary with the speeds of the respective particles according to the well-known law of special relativity. It is shown that the system of differential equations of motion can be solved by quadratures. The discussion of the physical implications of the solution is brief and incomplete. *L. A. MacColl.*

Manolov, Spasse. Sur l'existence des petits mouvements périodiques autour de la position d'équilibre relatif stable de pendules articulés, sous l'action d'une rotation uniforme. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1. **48** (1953/54), 1–16 (1954). (Bulgarian. French summary)

Manolov, S. On the existence of small periodic motions about a position of relative stable equilibrium of a mechanical system. Prikl. Mat. Meh. **19** (1955), 493–499. (Russian)

The second paper is an expansion of the first one. The system consists of a chain of n equal bars A_iA_{i+1} with hinges A_i ($i=1, \cdots, n$) horizontal and parallel; A_1 is moving uniformly on a horizontal circle (angular speed ω), and the axis of the hinge A_1 remains tangent to this circle. The Poincaré-Lyapunov perturbation method is applied. The equations of the first approximation are shown to be independent for $n \leqq 4$; the assumption is made that they so are for every n. The frequency equation, by application of the Sturm-chain criterion, yields then n distinct frequencies, all smaller than ω. For sufficiently small ω stability of the equilibrium position with all A_{i+1} vertically under A_i follows from Dirichlet's theorem, and the existence of periodic motions for the first approximation is established. It follows that the exact equations have periodic solutions in the neighborhood of the position of equilibrium for sufficiently small initial velocities. *A. W. Wundheiler* (Chicago, Ill.).

Gravina, Pedro B. J. Sulla teoria dei ponti sospesi e l'impiego delle funzioni d'influenza. Ann. Mat. Pura Appl. (4) **39** (1955), 203–227.

On the deflection theory (or second-order theory) of suspension bridges. The author starts from the Müller-Breslau equation and solves it by means of successive approximation. The solution is developed according to a new method in a trigonometric series. Considerations about influence functions. *O. Bottema* (Delft).

Boldinskiĭ, G. I.; and Zel'tin, A. I. On the motion of a system of variable mass. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **15** (1955), 93–98. (Russian)

The author considers a system of variable mass which is simultaneously losing mass by ejection and gaining it by adjunction. Starting from the general Lagrangian equations as given by A. A. Kosmodem'yanskiĭ [Moskov. Gos. Univ. Uč. Zap. **154**, Meh. **4** (1951), 73–180; MR **14**, 1025] he expresses them in various forms according to various assumptions regarding the dependence of the masses and velocities of ejection and adjunction on the generalized coordinates q_j, $\dot{q}_j$ and the time t, and the existence of cyclic coordinates. On p. 93 the definitions of m_{1i} and m_{2i} should presumably be interchanged. *R. A. Rankin.*

Speiser, Ambros P. Koordinatensysteme in Feuerleitgeräten. Z. Angew. Math. Phys. **7** (1956), 1–16.

See also: Mandel'štam, p. 969; Colombo, p. 969.

Fluid Mechanics, Acoustics

★**Dryden, Hugh L.; Murnaghan, Francis D.; and Bateman, H. Hydrodynamics.** Dover Publications, Inc., New York, 1956. 634 pp. $2.50.

Reprint by photo-offset of Bulletin no. 84 of the National Research Council, Washington, D.C., 1932.

Suharevs'kiĭ, I. V. On a boundary problem of hydrodynamics. II. Dopovidi Akad. Nauk Ukrain. SSR **1955**, 39–42. (Ukrainian. Russian summary)

The author proves a theorem concerning the construction of a continuous solution $u(s)$ of an integral equation connected with the boundary problem of finding the velocity of a plane flow, without separation, of an ideal incompressible fluid about a convex piece-wise smooth contour C ($z=z(s)$; $-l_2 \leqq s \leqq l_1$) with one angular point $z_0=z(0)$, for a discrete or continuous distribution of singularities in the region of the flow [see part I, same Dopovidi **1954**, 416–418; MR **17**, 547]:

$$u(s)=\frac{1}{\pi}\int_C u(\sigma)\frac{\partial \ln|\zeta-z|}{\partial n_\sigma}d\sigma=2\,\mathrm{Re}\,[w_0(z)z'(s)]$$

($w_0(z)$ is the complex velocity of the undisturbed flow, $\zeta=z(\sigma)$, n the outwards normal unit vector).

Let $\arg z'(s)$ satisfy a Lipschitz condition with positive exponent in the intervals $[-l_2, 0]$, $[0, l_1]$. Let C' be the arc of the convex contour C on which

$$\varepsilon_1 \leqq s \leqq l_1, \quad -\varepsilon_2 \geqq s \geqq -l_2,$$

$$K(\sigma, s)=\frac{1}{\pi}\frac{\partial}{\partial n_\sigma}\ln|\zeta-z|$$

(n the inwards normal unit vector). Then the following results hold. 1) The integral equation

$$u_1(s)-\lambda\int_{C'}K(\sigma, s)u_1(\sigma)d\sigma=0$$

does not have eigenvalues in the circle $|\lambda| \leqq 1$ and consequently the equation

$$u_1(s)-\int_{C'}K(\sigma, s)u_1(\sigma)d\sigma=f(s),$$

with $f(s)$ continuous on C', has a unique continuous solution $u_2(s)$. 2) If $f(s)$ is piece-wise continuous on C with the only possible point of discontinuity at z_0 and if the integral equation

$$u(s)-\int_C K(\sigma, s)u(\sigma)d\sigma=f(s)$$

has a continuous solution $u(s)$, equal to zero at the angular point, then, for $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\varepsilon_2/\varepsilon_1=O(1)$, $u_1 \to u$ uniformly on any arc of C not containing the angular point.

Yu. D. Sokolov (RŽMeh **1956**, no. 801).

Harlamov, P. V. Translational motion of a heavy rigid body in a fluid. Prikl. Mat. Meh. **20** (1956), 124–129. (Russian)

A rigid body moves through inviscid liquid otherwise at rest. The author investigates the conditions of translation without rotation in the case in which the weight of the body is equal to the weight of the liquid displaced but their centres of gravity are distinct, so that weight and buoyancy form a couple. The possible directions of permanent translation are found to be parallel to generators of a quadric cone. Various cases are discussed, the fundamental role being played by the position vector of

the centre of gravity of the body with respect to the centroid of the volume of the body.

L. M. Milne-Thomson (Greenwich).

Haskind, M. D. Unsteady motion of a rigid body in an accelerated flow of an unbounded fluid. Prikl. Mat. Meh. **20** (1956), 120–123. (Russian)

A rigid body moves with velocity $(\mathbf{V}(t), \mathbf{\Omega}(t))$ in inviscid liquid whose velocity at infinity is $\mathbf{V}_c(t)$. Write $\varphi=\Phi-\mathbf{r}\cdot\mathbf{V}_c$, where Φ is the velocity potential and let

$$\mathbf{K}=-\varrho\int\varphi\mathbf{n}dS,\quad \mathbf{L}=-\varrho\int\varphi(\mathbf{r}\wedge\mathbf{n})dS.$$

Then the force and moment due to the liquid are exhibited in the form

$$\mathbf{F}=-\partial\mathbf{K}/\partial t-\mathbf{\Omega}\wedge\mathbf{K}+M'd\mathbf{V}_c/dt,$$

$$\mathbf{M}=-\partial\mathbf{L}/\partial t-\mathbf{\Omega}\wedge\mathbf{L}-\mathbf{V}\wedge\mathbf{K}+\mathbf{V}_c\wedge\mathbf{K}+M'\mathbf{r}_c\wedge d\mathbf{V}_c/dt,$$

where M' is the mass of liquid displaced and $\mathbf{r}_c$ is the position vector of the centroid of the body. When $\mathbf{V}_c=0$ these reduce to the Kirchhoff vectors [Milne-Thomson, Theoretical hydrodynamics, 3rd ed., Macmillan, New York, 1956, 17.43; MR **17**, 796]. Thus the analysis effectively isolates the formal additions due to time dependence. *L. M. Milne-Thomson* (Greenwich).

★ **Squire, H. B. Rotating fluids.** Surveys in mechanics, pp. 139–161 (2 plates). Cambridge, at the University Press, 1956. $9.50.

This is one of ten articles constituting the G. I. Taylor 70th-anniversary volume. This article is a review of the major theoretical developments in the study of the motion of an ideal fluid which is assumed to rotate as a solid body in the undisturbed state. *G. W. Morgan.*

★ **Carafoli, Elie. Tragflügeltheorie (inkompressible Flüssigkeiten).** Verlag Technik, Berlin, 1954. 562 pp.

This is a translation from the Roumanian [Editura Tehnica, Bucureşti, 1951] by Norbert Wass. A very extensive coverage of straight wings (especially biplanes) in incompressible flow is presented, but the treatments of sweepback and unsteady flow are far from representative of present-day knowledge; compressibility is mentioned on the first page as a phenomenon to be neglected (the reviewer was unable to find any mention of the Prandtl-Glauert correction). Even within these limitations, coverage appears (to the reviewer) rather uneven; e.g., the work of Joukowsky is cited more often than that of Glauert, Munk, or Prandtl, and specific references to the literature, especially post-World War II, are even more spotty (one paper by Munk, two by Prandtl). The specialist may find the detailed treatment of many problems valuable, but the foregoing objections seriously detract from the value of this work as either a text or a treatise. *J. W. Miles* (Los Angeles, Calif.).

Guennoc, H. Calcul d'un profil symétrique en écoulement incompressible. Rech. Aéro. no. **50** (1956), 13–15.

Müftüoglu, O. Saffet. Die Geschwindigkeitsverteilung an der ebenen Platte im Gitterverband. Abh. Braunschweig. Wiss. Ges. **6** (1954), 220–226.

The velocity distribution on the flat plate in cascades is calculated by means of conformal mapping for values of the solidity $t/l=0.5$; 0.75; 1.0 and 1,5 and of angles of stagger $\lambda=0°$, 30°, 45° and 60°. *Author's summary.*

Gheorghiţă, Şt. I. Un problème plan de percussion, avec détachement. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 8, 67–71. (Romanian. Russian and French summaries)

The author obtains expressions for the impulse distribution and the velocity components for the motion of a plate (normal to its plane) by means of conformal mapping. *K. Bhagwandin* (Oslo).

Gheorghiţă, Şt. I. Sur le mouvement plan d'un fluide idéal, en présence d'une enveloppe poreuse. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 1013–1023. (Romanian. Russian and French summaries)

Saljnikov, Viktor. Influence of deformation of the bottom of a reservoir on the form of free jet. Univ. Beograd. Zb. Mašin. Fak. **1953-54**, 7–16. (Serbo-Croatian. French and English summaries)

Abbott, M. R. A theory of the propagation of bores in channels and rivers. Proc. Cambridge Philos. Soc. **52** (1956), 344–362.

In this paper a theory is presented of the nonlinear propagation of waves and bores in rivers (and channels) of varying cross-section with a basic steady flow governed by the balance between frictional and gravitational forces. The Chézy law is used to represent the frictional resistance, though the theory can easily be modified to adopt any friction law. Some methods used in the analogous shock-wave problems in gas dynamics are extended to treat this problem within the framework of shallow-water theory. First, the linear theory is stated, the type of solution which is adopted is such that the velocity and wave profile perturbations are expanded in series of successive integrals of an arbitrary function of the linear characteristic variable, with coefficients being functions of the distance along the river. Following the method due to G. B. Whitham [Comm. Pure Appl. Math. **5** (1952), 301–348; **6** (1953), 397–414; MR **14**, 330; **15**, 751], the linear theory is improved by replacing the linear characteristic variable by the more exact variable which includes the first-order perturbation quantities. The theory is thus developed to provide a condition on the tidal range required to produce a bore, in terms of the geometry and friction parameters of the river, and the propagation of such a bore is then described. The results for the special case of waves advancing into still water in a channel of varying section are deduced from the general case by setting both the friction and steady velocity equal to zero. Finally, the theory is applied to the well-documented case of the River Severn and the results agree fairly well with observation. *T. Y. Wu* (Pasadena, Calif.).

Long, Robert R. Long waves in a two-fluid system. J. Meteorol. **13** (1956), 70–74.

Dans l'hypothèse d'une distribution hydrostatique de la pression, l'auteur étudie la propagation des ondes d'amplitude finie à la surface de séparation de deux liquides de densités différentes, contenus entre deux parois planes, horizontales, distantes d'une quantité finie H. Ceci différencie ces mouvements de ceux considérés par Tepper dans un travail récent pour lesquels la couche supérieure était d'épaisseur infinie. Les équations du mouvement sont résolues par la méthode des caractéristiques.

Dans la région non atteinte par l'onde, chacun des deux

liquides est supposé animé d'un mouvement rectiligne uniforme de vitesse u_0 pour la couche inférieure, v_0 pour la couche supérieure. Soient, h la hauteur d'un point de la ligne de séparation au-dessus du plan inférieur, $R=h/H$, R_0 la valeur de R dans la région non troublée.

L'inégration des équations du mouvement montre que la vitesse de propagation du profil de l'onde est une fonction de R, R_0, u_0, v_0. Cette vitesse dépendant de R, le profil se déforme en se propageant. Pour quelques valeurs de R_0 et de u_0-v_0 l'auteur a construit les courbes donnant la vitesse de propagation en fonction de R. Ces courbes mettent en évidence que cette vitesse n'est pas une fonction monotone de l'amplitude. Lorsque $u_0=v_0=0$, une onde d'amplitude assez faible déferle sur l'avant si $R_0<\frac{1}{2}$ et sur l'arrière si $R_0>\frac{1}{2}$. Le déferlement sur l'avant (resp. sur l'arrière) est favorisé lorsque $v_0>0$ (resp. $v_0<0$). *R. Gerber* (Toulon).

Kubota, Keichi. An analytical theory on the seepage through an earth dam by the method on hodographs. Sci. Papers Fac. Engrg. Tokushima Univ. **2** (1950), no. 1, 13–21. (Japanese. English summary)

Loĭcyanskiĭ, L. G. On the theory of a ball bearing. Prikl. Mat. Meh. **20** (1956), 133–135. (Russian)

In this paper the author discusses more precisely some of the results obtained by him in a previous paper [Prikl. Mat. Meh., **19** (1955), 531–540; MR **17**, 679]. The expressions for the appropriate vector components are given in a simpler form. Also, it is shown that the expressions obtained by Wannier [Quart. Appl. Math. **8** (1950), 1–32; MR **12**, 217] are limiting cases of the present solutions. Evidently, the author does not seem to be familiar with Kuo's criticism (loc. cit.) of Wannier's paper.

K. Bhagwandin (Oslo).

Ionescu, Dan Gh. Intégrales générales des mouvements à symétrie axiale des fluides visqueux incompressibles. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **3** (1954), no. 4–5, 143–147. (Romanian. Russian and French summaries)

The author obtains some expressions (not altogether new) for the axial symmetrical flow of viscous incompressible fluids. Actually, the author obtains no closed-form solution to any specific problem. *K. Bhagwandin.*

Iacob, Caius. Sur quelques écoulements lents des fluides visqueux. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2** (1953), no. 3, 43–51. (Romanian. Russian and French summaries)

The author deals primarily with the Poiseuille type of viscous fluid flow through tubes of arbitrary cross-sections; the generatrix is taken parallel to the axis of the tube. The solution is given in terms of a Green's function. The Dirichlet problem is also considered. Special attention is paid to the case of an infinitesimal deformation of the base contour. Here the author resorts to properties of conjugate harmonic functions. The expressions for the discharge and the relative correction are stated. Particular cases as well as possible applications of the author's theory are noted. *K. Bhagwandin* (Oslo).

Gheorghiţă, Şt. I. Le mouvement stationnaire d'un fluide visqueux dans un tube cylindrique dont la section est voisine d'un rectangle. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2** (1953), no. 3, 62–65. (Romanian. Russian and French summaries)

The author determines approximately the stationary flow of viscous incompressible fluid through a tube of rhomboidal cross-section in terms of a Green's function. An approximate expression for the discharge is also given. *K. Bhagwandin* (Oslo).

Gheorghiţă, Şt. I. De la distribution des tourbillons dans un fluide visqueux se mouvant en présence d'un corps poreux. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 1003–1011. (Romanian. Russian and French summaries)

Bogdan, Gabriela. Relations entre les tensions dans un fluide visqueux incompressible dans le mouvement lent non permanent. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2** (1953), no. 3, 66–72. (Romanian. Russian and French summaries)

The author establishes some relations between the various tensorial components appropriate to the slow motion of viscous incompressible fluids. The method used consists of introducing two associated matrix differential operators [cf. G. C. Moisil, Associated matrices of systems of partial differential equations···, Editura Acad. R. P. Române, 1950; MR **16**, 1028]. This paper is not available to the present reviewer, and therefore it is not possible to state the author's actual contribution. *K. Bhagwandin* (Oslo).

Bloh, È. L. Flow of a viscous gas between two moving parallel cylindrical surfaces of arbitrary shape. Prikl. Mat. Meh. **20** (1956), 116–119. (Russian)

The gas and cylinders are in steady motion parallel to the generators with walls at constant temperatures T_1 and T_2. It is shown that the frictional drag F, on one of the cylinders, is related to the frictional drag F^*, in a corresponding problem in which the gas is replaced by incompressible fluid, by a formula $F=KF^*$. The constant K is independent of shape and is calculated in terms of T_1, T_2 and the Prandtl number. The method is illustrated by taking the cross-section enclosed by confocal ellipses.

L. M. Milne-Thomson (Greenwich).

Sanyal, Lakshmi. The flow of a viscous liquid in a circular tube under pressure-gradients varying exponentially with time. Indian J. Phys. **30** (1956), 57–61.

Lagerstrom, P. A.; and Cole, J. D. Examples illustrating expansion procedures for the Navier-Stokes equations. J. Rational Mech. Anal. **4** (1955), 817–882.

This lengthy paper is concerned with a study of approximation methods used in the theory of viscous incompressible flow. The particular problem that is studied is that of an infinitely long circular cylinder moving with a constant velocity U in the direction of its axis in a viscous fluid. At the same instant at which the cylinder starts to move ($t=0$), the radius of the cylinder starts to increase proportionally to t^n, $0\leq n\leq 1$. In the cases $n=0$ and $n=\frac{1}{2}$ the resultant problem can be solved exactly.

Variables appropriate to such approximate methods as those of Stokes, Oseen, the Prandtl boundary layer, and Euler are introduced, and limiting processes are defined. The exact solutions are then expressed in each of these sets of variables, and expansions are obtained by repeated application of the appropriate limiting process. Approximate solutions for $0\leq n\leq 1$ are derived by assuming that the solutions may be represented by ex-

pansions of suitable form; differential equations for the various terms are derived.

The question of uniformity of the expansions is discussed, and it is shown that an expansion which is uniformly valid for all r and t, (r=radius) can only be obtained in the case $n=\frac{1}{2}$. A general discussion of the nature of the Stokes and Oseen expansions and their relationship to one another is presented.

R. C. Di Prima (Cambridge, Mass.).

Jain, M. K. The motion of an infinite cylinder in rotating non-Newtonian liquid. Z. Angew. Math. Mech. **35** (1955), 379–381. (German, French and Russian summaries)

This note concerns Taylor's result [Proc. Roy. Soc. London. Ser. A. **93** (1917), 99–113] that the forces exerted on a cylindrical body moving in an incompressible fluid (plane flow) change only by a simple additive term and that the torque remains the same if a uniform rotation is superposed on the whole system. It is proved that this theorem remains valid for a special type of non-Newtonian fluid. [Reviewer's remark: Taylor's theorem is valid for any incompressible medium whatsoever, as can be seen without computations.] *W. Noll.*

Have

★**Lin, C. C. The theory of hydrodynamic stability.** Cambridge, at the University Press, 1955. xi+155 pp. $4.25.

This monograph surveys the field of hydrodynamic stability in its present theoretical form along with the relevant experimental results. The basic problem is formulated in terms of a given steady viscous flow and the linearized equations for the velocity differences of a time-dependent disturbance of this flow. These equations have solutions that satisfy the boundary conditions of the original flow and are exponential in time t. If the exponent has negative real part, the velocity differences die out as $t\to\infty$ and the flow is stable; if the exponent is negative, the velocity differences become large and the flow is unstable. Sufficient conditions for stability in many cases have been established rigorously but necessary conditions, to which the author has contributed extensively, have been harder to prove and have led to much controversy. The first difficulty is that a reasonable physical approach does not reveal immediately which role viscosity plays. Does its action of damping disturbances or of creating turbulence dominate? Secondly, a complete mathematical theory is far too complicated to analyze. Approximations must be made and their validity examined.

The plan of the book is to consider a few simple cases in detail and to sketch the more complicated cases. In Chapter 1 three basic problems of stability are formulated: the flow between parallel plates, the flow between rotating cylinders, and the flow through a circular pipe. An example of the first type of flow, plane Poiseulle motion, is discussed at length in Chapter 3. The flow between rotating cylinders (Couette motion) is discussed in Chapter 2 and a much simplified version of G. I. Taylor's analysis is presented. Sufficient conditions are also deduced.

Chapter 4 contains general discussions of the role of viscosity, the inviscid case including the correct approach to this limit, the connection of vorticity with stability and the transfer of energy. In addition there is a section on the qualitative effects of finite amplitude disturbances.

Boundary-layer stability is described in Chapter 5. There is a larger number of new independent variables and the system of equations to be studied is of higher order. The effects on stability of the temperature on the boundary and of the Mach number of the flow are discussed.

In Chapter 6 the transition to turbulence as opposed to the instability of the laminar flow is outlined. This is followed by a description, with experimental evidence, of the effects of cooling the boundary, changing the pressure gradient downstream, sucking fluid from or injecting fluid into the boundary layer, and changing the curvature of the boundary. The chapter ends with a section on the entry region in pipes and channels and another on the mixing of two parallel streams.

Chapter 7 deals with various other effects in nearly parallel flows. These include Coriolis forces, heat convectivity, gravitational and electromagnetic fields.

Chapter 8 deals with some of the mathematical aspects of the problem, particularly the asymptotic theory involved. The interesting phenomenon of friction layers and friction regions is analyzed. The book concludes with a very extensive bibliography.

This first up-to-date monograph on stability theory by an outstanding expert will be indispensable to all workers in the field. *C. S. Morawetz* (New York, N.Y.).

Steinman, H. The stability of viscous flow between rotating cylinders. Quart. Appl. Math. **14** (1956), 27–33.

In this paper Chandrasekhar's method of solving the eigenvalue problem [Mathematika **1** (1954), 5–13; MR **16**, 84] is applied to a more accurate formulation of the theory. Specifically, in an approximate expression for the steady velocity of rotation, terms of relative order of magnitude $(R_2-R_1)/R_1$ previously neglected are now retained, where R_1 and R_2 are the radii of the inner and outer cylinders, respectively. (However, corresponding improvements in accuracy are not made in the case of other approximations present in the basic equations and boundary conditions.) The present numerical calculations are in general closer to the experimental results of Taylor [Philos. Trans. Roy. Soc. London. Ser. A. **223** (1923), 289–343] than either Chandrasekhar's calculations or Taylor's original theoretical calculations. *D. W. Dunn.*

Lin, C. C. Aspects of the problem of turbulent motion. J. Aero. Sci. **23** (1956), 453–461, 516.

Expository paper.

★**v. Krzywoblocki, M. Z. On turbulence in rarefied gases.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 677–685. American Society of Mechanical Engineers, New York, 1955. $9.00.

The author derives the equations of the statistical theory of turbulence for a gas satisfying Grad's "thirteen-moment" equations of motion. *M. J. Lighthill.*

Vernotte, Pierre. Calcul du coefficient de diffusion à partir de mesures de diffusion en régime variable. Actes du colloque sur la diffusion, Montpellier, 1955, pp. 17–31; discussion 32. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. 59, Paris, 1956.

Filimon, I. Sur le mouvement subsonique à circulation des fluides compressibles autour d'un obstacle circulaire. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1953), no. 2, 38–43. (Romanian. Russian and French summaries)

C. Iacob [Com. Acad. R. P. Române **1** (1951), 741–746;

MR 17, 550] has described a method for constructing compressible flows at Mach number M_0 and speed U in the $z=x+iy$ plane that correspond to incompressible flows with circulation in the $\zeta=\xi+i\eta$ plane. To terms of order M_0^2 the correspondence is given by

$$(*) \qquad dz=d\zeta+M_0^2[ig(\zeta)d\zeta-(\overline{df/d\zeta})^2\overline{d\zeta}/4U^2],$$

$$(**) \qquad \omega=\theta+i\log V=\omega_0+\omega_1M_0^2.$$

Here $Ve^{i\theta}$ is the velocity in the compressible flow; f is the complex velocity-potential function for incompressible flow with prescribed speed and circulation about a given profile; g is an analytic function of ζ; $i\omega_0=-\log df/d\zeta$; and $\log 4iU^2(g-\omega_1)=-i(\omega_0-\overline{\omega}_0)$. Now let σ be arclength on the circle $|\zeta|=1$ and s arclength on its image in the z-plane. The author chooses f for flow about the circle and selects g so that $ds/d\sigma=1+O(M_0^4)$, thereby obtaining an approximately circular image in the z-plane. To terms of order M_0^2 this requires the determination of the analytic function $g-\zeta dg/d\zeta$ with prescribed imaginary part on $|\zeta|=1$. Since g becomes a quadratic polynomial in $1/\zeta$, the desired boundary and velocity distribution can be found immediately from (*) and (**).

J. H. Giese (Aberdeen, Md.).

Popp, Simona. Sur un problème concernant le mouvement d'un fluide compressible. I, II. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1953), no. 2, 32–37; **4** (1955), no. 8, 59–65. (Romanian. Russian and French summaries)

Chaplygin's method has been applied to transform the incompressible flow in a symmetrical plane jet of finite width impinging on an infinite wedge into subsonic compressible flow of a finite jet about the same wedge. The second paper extends the discussion to the asymmetric case.

J. H. Giese (Aberdeen, Md.).

Nocilla, Silvio. Sopra una classe di soluzioni singolari della equazione di Tomotika e Tamada per lo studio dei moti transonici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 55–61.

The author follows Tomotika and Tamada [Quart. Appl. Math. **9** (1951), 129–147; MR **13**, 179], who approximated the hodograph equation of plane irrotational transonic flow for the stream function ψ by

$$\tau^2\frac{\partial^2\psi}{\partial\tau^2}+\tau\frac{\partial\psi}{\partial\tau}+(1-\tau^2)\frac{\partial^2\psi}{\partial\beta^2}=0,$$

and obtained solutions

$$\psi_{-\frac{1}{2}}=\sum_{n=0}^{\infty}\lambda^{-n}J_n(n\tau)e^{-in\beta}=\frac{1}{1-\tau\cos\omega},\ \psi_{\frac{1}{2}}=\psi_{-\frac{1}{2}}d\beta=\omega,$$

where ω is a solution of the equation

$$\beta-i\log\lambda+\tau\sin\omega-\omega=0.$$

These solutions are specially useful as they have branch-point singularities of the form correct for the representation of a flow about a body.

The author first studies them in more detail as analytic functions of τ and of $\beta-i\log\lambda$, and the goes on to study functions $\psi_{3/2}$, $\psi_{5/2}$ and $\psi_{7/2}$ obtained by further integrations. These are used in the following paper. *M. J. Lighthill.*

Nocilla, Silvio. Sopra una classe di profili alari transonici nell'approssimazione di Tomotika e Tamada. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89** (1954–55), 296–322.

Using the functions derived in the preceding paper, the author develops a whole series of complete transonic flows past aerofoils, analogous to the single flow developed by Tomotika and Tamada in their paper cited in the preceding review. As in that case, the aerofoils have a pointed leading edge with semi-angle 18°. However, the author develops also a modification method, in which rather simple solutions of the hodograph equation are added on, which has the effect of making the leading edge blunter. These succeed only partially in making it look like an ordinary aerofoil leading edge.

M. J. Lighthill (Manchester).

Gras, François. Solutions trivalentes des équations réglant l'écoulement plan d'un fluide compressible. C. R. Acad. Sci. Paris **242** (1956), 991–994.

The author obtains a new three-valued solution of the hodograph equation for plane irrotational transonic flow. Such solutions are known [Lighthill, Proc. Roy. Soc. London. Ser. A. **191** (1947), 323–341; MR **9**, 391] to be useful for representing flows with branch lines such as flows through Laval nozzles.

M. J. Lighthill.

Kogan, M. N. On the theory of flow about bodies of small elongation. Prikl. Mat. Meh. **20** (1956), 87–94. (Russian)

Let s, r, θ be cylindrical coordinates. Let $r=f(s)$ be a pointed body of revolution for which f and df/ds are $O(\varepsilon)$. Let ϕ be a velocity potential function for linearized supersonic flow with $\phi=0$ on the Mach cone of the vertex and prescribed $\partial\phi/\partial r=\omega(s,r)$ on the body. Now consider a transformation $s_1=s$, $\xi=\xi^*(s,r,\theta)$, $\eta=\eta^*(s,r,\theta)$, where $\xi^*+i\eta^*$ is an analytic function of $re^{i\theta}$ such that $\xi^*+i\eta^*=re^{i\theta}[1+O(\varepsilon^2/r^2)]$. The author shows that to within terms of $O(\varepsilon^2\ln\varepsilon)$, which are comparable to the error involved in linearizing, ϕ satisfies

$$(M^2-1)\partial^2\phi/\partial s_1^2-\partial^2\phi/\partial\xi^2-\partial^2\phi/\partial\eta^2=0.$$

He proposes to apply this result to determine flows about slender objects, wings of low aspect ratio, etc. As an illustration he reduces the problem of finding the flow at an angle of attack α about a plane delta wing of semi-vertex angle arctan 2ε to that of finding the flow at angle of attack 2α about a cone of semi-vertex angle arctan ε. The ϕ so obtained agrees to within $O(\varepsilon^2\ln\varepsilon)$ with the velocity potential obtained directly from linearized conical-flow theory for the wing.

J. H. Giese (Aberdeen, Md.).

Sears, W. R. Some recent developments in airfoil theory. J. Aero. Sci. **23** (1956), 490–499.

Stewart, H. J. A review of source superposition and conical flow methods in supersonic wing theory. J. Aero. Sci. **23** (1956), 507–516.

Holt, M. The initial behaviour of a spherical explosion. I. Theoretical analysis. Proc. Roy. Soc. London. Ser. A. **234** (1956), 89–109.

Berry and Holt [same Proc. **224** (1954), 236–251; MR **16**, 303] treated the problem of the propagation of the spherical blast obtained from the detonation of a homogeneous spherical mass of explosive. Both the explosive gas and the disturbed medium were taken as polytropic and convenient values of the adiabatic constants were used. In this paper the same problem is considered for arbitrary equations of state for both the explosive gas and the surrounding medium. Essentially the same procedures

as developed in the previous paper are used and it is shown that the properties established for the polytropic case also apply for this more general situation. Specific formulas are obtained describing the initial behavior of the blast in terms of the initial parameters.

P. Chiarulli (Washington, D.C.).

★ **Lighthill, M. J. Viscosity effects in sound waves of finite amplitude.** Surveys in mechanics, pp. 250–351 (2 plates). Cambridge, at the University Press, 1956. $9.50.

Cet article est une étude bibliographique remarquable sur les mouvements des fluides compressibles visqueux et conducteurs de chaleur. Pour commencer, les équations du mouvement sont établies et les lois physiques qui régissent les phénomènes de viscosité et de conductivité thermique sont énoncées. L'atténuation et la dispersion des ondes sonores est étudiée sur les équations obtenues en linéarisant les termes appelés „termes de diffusion" qui traduisent les phénomènes précédents; les formules théoriques sont comparées avec les résultats expérimentaux relatifs au gaz carbonique, à l'azote et à l'oxygène.

L'auteur aborde ensuite la théorie classique de la formation des ondes de choc dans les fluides thermodynamiquement réversibles (c'est-à-dire à entropie constante); l'introduction de la viscosité remplace la discontinuité de vitesse par une zone dans laquelle le gradient de la vitesse prend de grandes valeurs; l'étude de cette zone fait connaître la structure de l'onde de choc. L'étude des ondes de choc est reprise ensuite sur l'équation approchée de Burgers [Advances in applied mechanics, Academic Press, New York, 1948, pp. 171–199; MR **10**, 270]; cette équation peut être écrite de la façon suivante:

$$\frac{\partial u}{\partial t}+2u\frac{\partial u}{\partial x}=\varepsilon\frac{\partial^2 u}{\partial x^2},$$

où u désigne la moitié de la vitesse, t le temps, x l'abscisse et ε un paramètre nul dans le cas des écoulements réversibles; la solution générale obtenue par Hopf [Comm. Pure Appl. Math. 3 (1950), 201–230; MR 13, 846] est indiquée. Lorsque ε est nul, la solution $u(t, x)$ peut être interprétée comme la valeur de u qui, pour t et x donnés, rend minimum la quantité $F=-ux+u^2t+f(u)$, où la fonction $f(u)$ est arbitraire. L'auteur utilise une quantité analogue à F pour déterminer par des égalités d'aires la trajectoire du choc; on obtient les mêmes résultats en remarquant, avec Paul Germain, que la valeur de u qui rend F minimum strict vérifie, non seulement l'équation de Burgers (avec $\varepsilon=0$), mais encore, lorsqu'elle est discontinue, la condition de choc qui s'écrit ici $U=u_1+u_2$, où U est la vitesse de déplacement de l'onde de choc, u_1 et u_2 les valeurs de u avant et après le choc.

L'auteur étudie ensuite la structure des chocs non stationnaires et, en particulier, la rencontre de deux chocs; il montre comment l'équation de Burgers permet de traiter des problèmes relatifs aux ondes de choc non planes. Il termine par une étude des effets de relaxation qui interviennent (par opposition aux effets de diffusion) dans les mouvements qui ne peuvent plus être considérés comme voisins d'un état d'équilibre donné; ces effets ont donné naissance à la théorie de la dispersion des ondes de choc de Bethe et Teller [Cornell University, (1941)= Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. X 117]. *H. Cabannes.*

Williams, W. E. Difffraction by a cylinder of finite length. Proc. Cambridge Philos. Soc. **52** (1956), 322–335.

The author's summary is as follows. "The diffraction of a plane harmonic sound wave by a hollow circular cylinder of finite length is considered. The problem is treated by using Laplace transforms and is reduced to the solution of two complex integral equations. An approximate solution is obtained for these equations when the product of the wave number (k) and the cylinder length (l) is large. The resonance of the system is considered and an equation derived for the resonant lengths which is then solved approximately for kl large. An explicit expression is obtained for the end correction of an r-resonant system (i.e. $kl\sim r\pi$), and also comparison is made with experimental results." *A. E. Heins* (Pittsburgh, Pa.).

Rytov, S. M. Acoustic properties of a finely laminated medium. Akust. Ž. **2** (1956), 71–83. (Russian)

Expressions are derived for the plane progressive compressional and shear wave velocities c_{zz}, c_{xx}, c_{xz} ($=c_{zx}$) and c_{xy} in an infinite medium consisting of very thin alternating plane layers of two isotropic materials stacked up along the z-axis. Elastic moduli are obtained therefrom, and it is indicated that the elastic properties of such a medium are like those of a crystal having hexagonal symmetry. Cases discussed include both layers of solid materials, one layer solid the other a viscous fluid, one layer solid the other an ideal fluid. Decay constants are obtained by including volume and shear viscosities, as appropriate. *W. W. Soroka.*

See also: Ippen, p. 931; Eminton and Lord, p. 1008; Verschaffelt, p. 1017; Anglès d'Auriac, p. 1024; Birkhoff and Diaz, p. 1030.

Elasticity, Plasticity

★ **Anglès d'Auriac, Paul. Introduction à la mécanique des milieux continus.** Publ. Sci. Tech. Ministère de l'Air, Paris, no. 306 (1955), ix+139 pp. 1300 francs.

This monograph is a very clear introductory text, excellently suited for students with mathematical as well as with physics or engineering interests. It is concerned with the systematic analysis of the general concepts of the subject and with the derivation of its fundamental equations rather than with methods to solve special problems for special bodies.

The first four chapters are devoted to the kinematics of deformation. The following two deal with the notion of stress and contain the derivation of Cauchy's laws of motion. The investigation of special bodies is confined to perfect and linear viscous fluids (chapter VII) and to elastic solids according to the linearized theory (chapters VIII–XI). The subject matter is very well organized, illustrated with 57 figures, and supplemented by a chapter with 12 problems on kinematics.

The mathematical instrument used is the concept of a linear transformation and its representation by a matrix. The fear of the author that his avoiding the explicit use of tensor calculus might displease the mathematically minded readers seems quite unjustified to the reviewer. On the contrary, the intrinsic notion of a linear transformation is suited to the subject at least as well as the equivalent notion of a second-order tensor. It would have

added to the elegance of the treatment if the author had made even more use of the algebra and the calculus of linear transformations rather than to descend always to their matrix representation in a Cartesian coordinate system. But it is possible that such a procedure would have impaired the usefulness of the book for readers with only little mathematical training. The definition of the stress tensor as a linear transformation given in Chapter V is the most natural one there is, and the author has no reason to excuse it as an "artifice".

This book continues the unfortunate French tradition of not including an index. *W. Noll* (Los Angeles, Calif.).

Post, E. J. Note about finite strain in elastic bodies. Physica **22** (1956), 243–246.

The author proposes a strain tensor, claimed to describe the change of distance in finite strain, in terms of a complicated Lie-Taylor series. He believes that the usual Cauchy deformation tensor, which he attributes to Brillouin, is appropriate only for rectilinear displacements. [Reviewer's comment: The author confuses the displacement-vector field of a finite deformation with the velocity field of a steady motion. His formula expresses only the Cauchy tensor of the deformation produced by an analytic steady motion in terms of its velocity field.] *W. Noll* (Los Angeles, Calif.).

Adkins, J. E. Cylindrically symmetrical deformations of incompressible elastic materials reinforced with inextensible cords. J. Rational Mech. Anal. **5** (1956), 189–202.

The bodies in consideration are assumed to exhibit curvilinear aeolotropy related to a cylindrical coordinate system r, θ, z in the undeformed state and to be reinforced with thin layers of inextensible cords lying in helical paths on cylindrical surfaces $r=$const. The type of deformations considered is the same as the one investigated by the author recently for unreinforced bodies [Proc. Roy. Soc. London. Ser. A. **229** (1955), 119–134; **231** (1955), 75–90, MR **16**, 973; **17**, 315]. It is shown that such deformations are possible also if the body is reinforced by up to four layers of cords. The number of parameters which specify the deformation, five for an unreinforced body, reduces by one with each layer of cords. The following simple deformations are included as special cases: inflation and extension of a cylindrical tube, flexure and extension of a curved block, and (as a limiting case) shear and flexure of a cuboid. The method of approach is the same as was used by the author and R. S. Rivlin [Philos. Trans. Roy. Soc. London. Ser. A. **248** (1955), 201–223; MR **17**, 427] in the special case of isotropic bodies.

W. Noll (Los Angeles, Calif.).

Grioli, Giuseppe. Limitazioni per lo stato tensionale di un qualunque sistema continuo. Ann. Mat. Pura Appl. (4) **39** (1955), 255–266.

This paper continues the exploitation of Signorini's theory of mean values of the stress in a continuous medium [Ann. Scuola Norm. Sup. Pisa (2) **2** (1933), 231–251, and later papers]; like an earlier paper of the author [Ann. Mat. Pura Appl. (4) **33** (1952), 239–246; MR **14**, 514], it employs the weighted means $p_{rst} \equiv \overline{t_{rs} P_t}$, where the P_t are a set of n functions orthogonal over the body. The author proves that

$$|t_{rs}|_{\max} > V \frac{|\sum_{q=1}^{n} a_q p_{rsq}|}{\int_V |\sum_{q=1}^{n} a_q P_q| dV},$$

where the a_q are arbitrary constants. He proves also that for a given body there exists a choice of the constants a_q which renders this new inequality a sharper estimate than his previously derived one.

For $n=3$, the author derives a particularly simple lower bound for the tension in a prism. He then chooses a_1, a_2, a_3 so as to maximize this bound and discusses the meaning of the result. There are also applications to bending. *C. Truesdell* (Bloomington, Ind.).

Bordoni, Piero Giorgio. On the exact relation between the the specific heats of an elastic solid. J. Rational Mech. Anal. **4** (1955), 975–981.

The author believes that the usual proof for the relation between the specific heats at constant volume and at constant pressure is unsatisfactory in the case of solids. He proves rigorously that the relation is valid for every natural state of a homogeneous isotropic elastic body possessing an essentially arbitrary isothermal strain energy function. (The reviewer believes that the usual proof is correct for any material as long as it obeys an equation of state relating volume, pressure, and temperature. The bodies considered by the author certainly satisfy such an equation for all states for which the stress reduces to a uniform pressure, as is clear from his equations (12) and (13). Thus the relation in question is valid for all states of hydrostatic stress, and not only for all natural states with zero stress.) *W. Noll.*

Iacob, Caius. Sur la torsion des barres cylindriques. Science R. P. Roumaine **1** (1953), 5–13 (1954).

French version of an earlier Romanian paper [Acad. R. P. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4** (1952), 669–677; MR **15**, 580].

Cadambe, V.; and Kaul, R. K. Torsional rigidity of narrow bars and tubes of twisted shape. J. Sci. Indust. Res. **13B** (1954), 673–677.

Abbassi, Mohammed M. Simple solutions of Saint-Venant torsion problem by using Tchebycheff polynomials. Quart. Appl. Math. **14** (1956), 75–81.

Known results are obtained by separating Laplace's equation in polar coordinates. *R. C. T. Smith.*

Capriz, Gianfranco. Sul problema di Saint Venant per i solidi elastici anche incomprimibili. Rend. Mat. e Appl. (5) **13** (1955), 495–506.

A direct treatment of St. Venant's problem in the classical linear theory of elasticity for incompressible bodies. *C. Truesdell* (Bloomington, Ind.).

★ **Tyler, C. M., Jr.; and Rouleau, W. T. An Airy integral analysis of beam-columns with distributed axial loading that deflects with the column.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 297–305. American Society of Mechanical Engineers, New York, 1955. $9.00.

The deflection, bending moment and stability of beam-columns having partially uniformly distributed axial loading which deflects with the member but whose line of action remains parallel to its undeflected direction are determined. The solution involves Airy integral functions, their first derivatives, and first integrals. The member is assumed straight and of uniform stiffness. Numerical calculations for several simple beam loadings are given and values of the required integrals of the Airy

Integral Functions needed for computations are tabulated. *W. Freiberger* (Providence, R.I.).

Buzin, E. I. On the form and stresses of a flexible elastic surface of revolution in equilibrium under the action of a given load. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 39–54. (Russian)

Membrane theory is developed for surfaces of revolution with rotationally symmetric stress systems, giving two equations of equilibrium and one of compatibility for the two non-zero stresses N_1, N_2. The necessary forms of the surface for these equations to be consistent are determined for two cases: (i) constant normal pressure; (ii) centrifugal force, the shell being of constant mass per unit area.

R. C. T. Smith (Cambridge, Mass.).

Nazarov, O. O. Solution of the differential equations of equilibrium of a sloping shell with given rectangular plane projection by the method of simple trigonometric series. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 13–19. (Ukrainian. Russian summary)

The author considers an open shell of small curvatures with given rectangular plane projection, of a constant thickness h, and given principal curvatures. The shell is supported on the edges and the normal load is expressed in Fourier series. The author solves the equations of equilibrium using simple trigonometric series, but the determination of coefficients for this series leads to a system of equations whose solution is not simple. The paper contains numerical tables with coefficients and displacements for several particular cases. *T. Leser.*

Korolev, V. I. Thin two-layered plates and shells. Inžen. Sb. **22** (1955), 98–110. (Russian)

A nearly exact theory of two-layered shells and plates was developed E. I. Grigolyuk [Inžen. Sb. **17** (1953), 69–120; MR **16**, 540], where the contact surface between the two layers is used as a reference surface.

The author of this paper presents an approximate theory of two-layered shells and plates where instead of the contact surface the neutral surface is used as a reference surface. In this approximate theory the flexural rigidity, the torsional rigidity, and the Poisson ratio for each layer are replaced by reduced effective rigidities and an effective ratio (which are specially weighted averages) applicable to a shell or plate as a whole. Equations determining tensile and shearing forces and flexural and torsional moments in uniform shells and plates (one-layered) become valid for two-layered shells and plates when these reduced effective rigidities and the effective ratio are introduced. Components of stress are determined through the internal forces and moments but the relations are not as simple as it is in the case of uniform shells and plates. Critical loads for two-layered shells and plates can also be found from the same formulas which were derived for uniform shells and plates when the rigidities and the Poisson ratio are replaced by the effective ones. The above method does not apply though to thermal stresses in two-layered shells and plates which are investigated separately by the author in a special paragraph.

The author solves numerically several cases by the exact method and by the approximate method and the agreement is amazingly close.

The approximate method is applicable (introduces only a negligible error) under the following conditions: a) when both layers are isotropic and of constant thickness and the contact surfaces cannot slide against each other; b) when the Love-Kirchhoff hypothesis for thin-walled structures is valid, that is when the ratios: $E_1\delta_1^2/E_2\delta^2$ and δ/a are negligible compared with unity, where E_1 and E_2 are coefficients of elasticity for the upper and lower layer respectively, δ_1 is the thickness of the upper layer, δ is the total thickness, and a is a characteristic dimension of a shell or plate. *T. Leser* (Aberdeen, Md.).

Buzin, E. I. On stresses in a plane plate according to the membrane theory. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 26–38. (Russian)

The author considers the question, whether generalized plane stress of a thin plate with body forces can be regarded as a limiting case of membrane theory of a shell, portion either of a cylinder or a sphere. No satisfactory physical correspondence can be obtained.

R. C. T. Smith (Cambridge, Mass.).

Sharma, Brahma Dev. Stresses in an infinite slab due to a nucleus of thermoelastic strain in it. Z. Angew. Math. Mech. **36** (1956), 75–78.

An infinite slab of uniform thickness is made of homogeneous, isotropic, elastic material. One face of the slab is affixed to a rigid base. Initially, the slab is at a uniform temperature. The author gives an analysis for the determination of the steady stress field set up when the temperature of a small element of the slab is raised. Numerical results illustrate the variation of the maximum normal stress at the rigid base with the position of the thermal element. *H. G. Hopkins* (Sevenoaks).

Wu, Ching-Sheng. Bending of rectangular plate with clamped edges. J. Aero. Sci. **23** (1956), 601–602.

Bogunović, Vladimir. On the bending of a rectangular plate with one edge free. Univ. Beograd. Zb. Građevin. Fak. **1952-53**, 129–186. (Serbo-Croatian. English and German summaries)

The proposed solutions apply to the bending of plates with one edge free and the following edge conditions: 1) Simply supported along three edges; 2) built in along the edge opposite to the free edge and the other two edges simply supported; 3) simply supported along the edge opposite to the free edge and the other two edges built in; 4) all three edges built in. In the first two cases the plate is arbitrarily loaded, while in the latter two it is loaded symmetrically with respect to the free edge axis of symmetry. *From the English summary.*

Minasyan, R. S. On a mixed problem in bending of a rectangular plate. Akad. Nauk Armyan. SSR. Dokl. **22** (1956), 3–12. (Russian. Armenian summary)

The author considers the bending of a thin rectangular plate under an arbitrary distribution of normal load. One pair of edges is simply supported and the other pair clamped over a portion of their length, simply supported over the remainder. The problem is reduced to solving a completely regular system of equations in infinitely many unknowns. The solution is unique and may be obtained by successive approximations. *R. C. T. Smith.*

Cadambe, V.; Kaul, R. K.; and Tewari, S. G. Flexure of thin elastic plates under specified edge tractions. Indian J. Phys. **29** (1955), 403–416.

In terms of the so-called "quasi-displacements" Southwell [Quart. J. Mech. Appl. Math. **3** (1950), 257–270; MR **12**, 372] formulated differential equations and boundary

conditions for flexural problems of thin plates. In this paper these are rederived by means of the principle of minimum complementary energy. Two examples are worked out by using the results. *Y. Y. Yu.*

Holmyanskiĭ, M. M. On traverse bending of thin elastic plates in the form of a circular sector. Inžen. Sb. **22** (1955), 199–205. (Russian)

If $2n$ normal loads of strength ± 1 are applied to a thin circular disc at suitable points, then, by symmetry, simply supported edge conditions hold along the straight edges of a circular sector of angle π/n. Consequently, influence functions for a sector can be derived from the corresponding functions for a complete disc. Simple closed expressions for influence functions for a circular sector with a clamped, simply supported, or free curved edge are deduced from results of A. I. Lur'e [Z. Prikl. Mat. Meh. (N.S.) **4** (1940), no. 1, 93–102]. *R. C. T. Smith* (Cambridge, Mass.).

★**Yu, Yi-Yuan. The influence of a small hole or rigid inclusion on the transverse flexure of thin plates.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 381–387. American Society of Mechanical Engineers, New York, 1955. $9.00.

Complex variable methods are used. The boundary of the hole is the ovaloid $z=K(e^{i\phi}+me^{-i\phi}+ne^{-3i\phi})$, where K, m, n are arbitrary real constants. Other authors have considered circular and elliptic holes and their results can be deduced as special cases. *R. C. T. Smith.*

Kononenko, T. I. Large bending of thin nonhomogeneous anisotropic plates. Har'kov. Gos. Univ. Uč. Zap. **39**, Trudy Fiz. Otd. Fiz.-Mat. Fak. 3 (1952), 95–100. (Russian)

Thin plates with very general linear stress-strain relations are considered. The only restriction on the elastic moduli is that $p_{13}=p_{23}=p_{33}=0$ implies $u_{13}=u_{23}=0$, where x_1, x_2, x_3 are rectangular cartesian axes, x_3 normal to the plate. The displacements u_1, u_2 are assumed to be linear in x_3, $u_3=\phi(x_1,x_2)$ may be large and second-order terms in are included in expressions for the strains. The strain energy is calculated and a generalization of the von Karman large-deflection equations obtained by considering its variation. *R. C. T. Smith*

Svirskiĭ, I. V. Modification of Galerkin's method for solution of a nonlinear problem of snapping of a bent plate. Inžen. Sb. **22** (1955), 42–47. (Russian)

The author gives a modification of Galerkin's method suitable for systems of the form $A(w)-P_k=0$, $A'(w)\delta w=0$ where A is a non-linear operator on w,

$$A'(w)\delta w=\partial A(w+\mu\delta w)/\partial\mu|_{\mu=0}.$$

Such a system occurs in the theory of buckling of a flat plate, first bent, then subjected to uniform pressure over the convex face. In this case P_k would be the critical load, w the deflection of a typical point in the middle surface. *R. C. T. Smith* (Cambridge, Mass.).

★**Sternberg, E.; and Eubanks, R. A. On the singularity at a concentrated load applied to a curved surface.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 237–245. American Society of Mechanical Engineers, New York, 1955. $9.00.

In this paper the authors study the nature of the singularity at the point of application of a concentrated load acting perpendicular to the curved boundary of an elastic body. In the neighborhood of the point of application of the load the bounding surface Σ is assumed to be sufficiently smooth and axially symmetric, the axis of symmetry coinciding with the load axis. If Σ is locally analytic, it is shown that the singularity is identical with that due to a concentrated load acting on the plane boundary of a half space (the Boussinesq singularity), if and only if the curvature of the meridian section of Σ vanishes at the load point. In case the curvature and rate of change of curvature do not both vanish at the load point, singularities of lower order than the Boussinesq singularity are still present in the first residual problem (the difference between the solution to the problem in question and the corresponding Boussinesq solution). The authors determine in closed form the lower order singularities required to reduce the first residual problem to one for which the surface tractions are finite and continuous. *L. E. Payne.*

Čankvetadze, G. G. Elastic half-space with a spherical cavity. Inžen. Sb. **22** (1955), 65–73. (Russian)

Axi-symmetric stress systems derived from a stress function with a number of arbitrary parameters are considered. By adjusting the parameters it is possible to satisfy approximately boundary conditions such as: the plane boundary free from external forces and the cavity subjected to uniform pressure. Inversion is used to transform the boundaries into two concentric spheres. *R. C. T. Smith* (Cambridge, Mass.).

Conway, Harry Donald. The indentation of a transversely isotropic half-space by a rigid punch. Z. Angew. Math. Phys. **7** (1956), 80–85.

Pîrvu, A. Sur un problème d'élasticité anisotrope. Acad. R. P. Romîne. Fil. Iași. Stud. Cerc. Ști. **6** (1955), 65–68. (Romanian. Russian and French summaries)

Inada, Sadatoshi. On the lateral vibrations of rods of exponentially variable cross section under tension or end thrusts. Sci. Papers Fac. Engrg. Tokushima Univ. 2 (1950), no. 1, 34–40. (Japanese. English summary)

★**Boyce, W. E.; Di Prima, R. C.; and Handelman, G. H. Vibrations of rotating beams of constant section.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 165–173. American Society of Mechanical Engineers, New York, 1955. $9.00.

Natural frequencies of a beam, rotating at a constant angular velocity about an axis at one end and vibrating transversely to the plane of rotation, are determined by approximate methods. Upper bounds are determined by the Rayleigh-Ritz and lower by the Southwell technique. A thorough mathematical discussion of the associated minimum principles is given and the role played by natural boundary conditions recognized. Various boundary conditions at the restrained end are considered and use of a simple polynomial trial function gives close bounds for the first two frequencies. *W. Freiberger.*

Pailloux, Henri. Sur les vibrations latérales d'une poutre chargée. C. R. Acad. Sci. Paris **242** (1956), 2097–2099.

Ekstein, H.; and Schiffman, T. Free vibrations of isotropic cubes and nearly cubic parallelepipeds. J. Appl. Phys. **27** (1956), 405–412.

In applying the Ritz method to elastic-vibration

problems the polynomial trial functions are often selected purely for their simplicity. The author obtains better approximations by choosing functions which are similar to actual solutions. Some are obtained as solutions of physically related problems, others by semidirect methods of the calculus of variations. Such trial functions satisfy neither the differential equations nor the boundary conditions of the given problem nor are they mutually orthogonal, but are chosen simple enough to be manageable. The frequency equation of 15th degree resulting from the set of simultaneous linear equations is reduced by group-theoretical methods and solved analytically. Also, the modes are classified by group theory according to symmetry properties and certain unexpected degeneracies found. *W. Freiberger* (Providence, R.I.).

★ **Miklowitz, Julius. Traveling compressional waves in an elastic rod according to the more exact one-dimensional theory.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 179–186. American Society of Mechanical Engineers, New York, 1955. $9.00.

In accordance with the more exact one-dimensional theory of compressional waves in an elastic rod due to Mindlin and Herrmann [Proc. 1st U.S. Nat. Congress Appl. Mech., Chicago, 1951, Amer. Soc. Mech. Engrs., New York, 1952, pp. 187–191], which takes into account the effect of radial inertia and radial shear deformation, the problem of a semi-infinite rod and that of a finite rod, both subjected to a sudden axial stress applied at one end, are solved by the Laplace-transform method. In the Mindlin-Herrmann theory the limiting phase velocity of the second mode for very short waves in an infinite rod does not agree with the results of Pochhammer's exact theory [A. E. H. Love, Theory of Elasticity, Cambridge Univ. Press. London, 1927, 4th ed., p. 287]. The finite-rod solution, which is presented in this paper and from which numerical calculation will be made, together with experimental measurements to be carried out, is expected to give further indication of the accuracy of the Mindlin-Herrmann theory. *Y. Y. Yu* (Syracuse, N.Y.).

Vorovič, I. I. On the behavior of a circular plate after buckling. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 55–60. (Russian)

The author considers an annular plate of constant rigidity and thickness. The normal and lateral loads are axisymmetrical and the author considers only symmetrical cases of buckling. The boundaries of the plate rest on non-linearly elastic supports. The author derives a procedure for finding approximately the condition of the plate after the loss of stability. The method permits one to establish the exact number of solutions, that is, all the possible forms of buckled equilibrium. The method is illustrated on an example of a circular plate on an elastic foundation whose reaction is proportional to the cube of the normal displacement, and the plate is compressed by a constant lateral load. *T. Leser* (Aberdeen, Md.).

★ **Lorsch, Harold G.; and Freudenthal, Alfred M. On mixed boundary value problems of linear viscoelastic solids.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 539–545. American Society of Mechanical Engineers, New York, 1955. $9.00.

This paper concerns plane infinitesimal strain of linear visco-elastic materials of Kelvin and Maxwell type. It is noted that time independent stresses which produce plane strain in the elastic case have not the same effect for visco-elastic bodies. Additional time dependent stresses normal to the plane of strain must be applied in order to maintain plane strain at all times. It is shown also that there is a similiar situation in the case of plane strain produced by a line load moving on the surface of a semi-infinite Maxwell body. Inertia effects are neglected throughout. *W. Noll* (Los Angeles, Calif.).

Cristescu, N. Les discontinuités dans le mouvement du fil parfaitement flexibile et viscoélastique. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **1** (1952), no. 1, 68–71. (Romanian. Russian and French summaries)

The author distinguishes three different types of discontinuities relative to the motion of a perfectly flexible visco-elastic thread. *K. Bhagwandin* (Oslo).

Duwez, Pol. Physics of solids — plastic flow. J. Aero. Sci. **23** (1956), 435–437, 468.

Velasco de Pando, Manuel. On a minimum principle in elastic-plastic states. Rev. Ci. Apl. **10** (1956), 97–100. (Spanish)

Drucker, D. C. On uniqueness in the theory of plasticity. Quart. Appl. Math. **14** (1956), 35–42.

The uniqueness problem in plasticity assumes an existing state of stress and displacement to be given throughout the body, and a suitable combination of traction rates and velocities to be given on the surface. It is then desired to show that the stress rates and velocities are unique throughout. The author proves uniqueness for elastic-perfectly plastic and elastic-work hardening materials by two methods. One is the conventional one which assumes two solutions with differences $\Delta\dot{\sigma}_{ij}$, $\Delta\dot{\varepsilon}_{ij}$. The principle of virtual work is then used to show that $\int_V \Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij}=0$, while according to the stress-strain rate law it is shown that $\Delta\dot{\sigma}_{ij}\Delta\dot{\varepsilon}_{ij}$ is positive definite. It follows that at least one of $\Delta\dot{\sigma}_{ij}$, $\Delta\dot{\varepsilon}_{ij}$ vanishes and hence uniqueness is proved.

The alternative approach is based directly upon the mechanical-thermodynamical postulates which are assumed to govern the behavior of plastic material. These postulates consider an external agency which applies and removes a set of forces to an already loaded body. During the application the work done by the external forces is positive, and during the complete cycle it is positive unless the material is perfectly plastic or the strain rates are only elastic, in which case it may be zero. It follows immediately from these postulates that any achieved solution is stable (at least when boundary motions are neglected). The author then shows that stability implies uniqueness for an elastic-perfectly plastic material, and for an elastic-work hardening material satisfying a linear or piecewise linear flow theory. In a final section, he presents a counter example in which the material obeys a law which does not satisfy the mechanical-thermodynamical postulates and the solution is not unique.

P. G. Hodge, Jr. (Brooklyn, N.Y.).

Mikeladze, M. Š. On plastic flow of anisotropic shells. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 8, 67–80. (Russian)

Hill's general theory of plastic anisotropy [The mathematical theory of plasticity, Oxford, 1950, Ch. XII; MR

12, 303] is specialized to the case of a thin shell of material in equilibrium. In the usual way, the fundamental equations are expressed in terms of stress-resultants and moments and incremental strains and curvatures, all of these quantities being defined in the shell middle surface. The general analysis is applied to determine the load-carrying capacity of a circular cylindrical shell, and also of a spherical shell, under internal pressure. Discussion is also given of built-up (i.e. multi-layered) shells in which the degree of anisotropy varies across the thickness. The load-carrying capacity is determined for a doubly-layered, circular cylindrical shell under internal pressure. The paper concludes with some general analysis in which the yield criterion is made dependent upon the hydrostatic pressure. *H. G. Hopkins* (Sevenoaks).

See also: Rytov, p. 1024; Biot, p. 1035.

MATHEMATICAL PHYSICS

Proca, A. Sur la mécanique spinorielle du point chargé. J. Phys. Radium (8) **17** (1956), 81–82.

The author notes that the equations he calls the equations of spinor mechanics [Nuovo Cimento (10) **2** (1955), 962–971; MR **17**, 558] are particularly simple in the cases he interprets as corresponding to a constant charge. This case is discussed briefly in this paper. *A. H. Taub.*

Proca, A. Sur un nouveau principe d'équivalence suggéré par les mécaniques spinorielles. J. Phys. Radium (8) **17** (1956), 83–84.

The author points out that if the definition of charge used in the equation of spinor mechanics obtains [Proca, Nuovo Cimento (10) **2** (1955), 962–971; MR **17**, 558] there is a relation between charge and the magnitude of the spin tensor. This relation and its interpretation is discussed. *A. H. Taub* (Urbana, Ill.).

See also: Craig, p. 1002.

Optics, Electromagnetic Theory, Circuits

Picht, Johannes. Über Reflexionen am Paraboloidspiegel. Wiss. Z. Pädagog. Hochsch. Potsdam **2** (1955/1956), 19–33.

Karp, Samuel N. Diffraction by an infinite grating of arbitrary cylinders. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-85** (1955), i+28 pp.

The author's summary is as follows. "Diffraction by an infinite grating of congruent arbitrarily shaped conducting cylinders is shown to be related to single scattering by an isolated typical grating element, even when all interactions among elements are taken into account. Fourier amplitudes of the diffracted field are given in terms of differential scattering amplitudes of the single cylinder: the latter are either analytically calculable (e.g., for the circular cylinder) or obtainable by experiment. The solution is approximate, holding under the assumption of large spacing as compared with both wavelength and cylinder dimensions." *A. E. Heins.*

Poincelot, Paul. Solution rigoureuse d'un phénomène particulier de diffraction. Ann. Télécommun. **11** (1956), 50–56.

A plane sinusoidal electromagnetic wave is incident normally upon a plane infinitely conducting rectangular screen, which is of infinite extent in one of its two dimensions. The electric field of the incident wave is parallel to the edges. The author obtains the vector potential of the field at the surface of the screen in the form of an infinite series of contour integrals, the coefficients of which satisfy an infinite system of algebraic equations in infinitely many unknowns. He discusses the convergence of the series and proves that the coefficients have the boundedness property of Fourier coefficients. He suggests that his method, i.e., reduction to an infinite algebraic system, may be applicable to conducting screens of arbitrary shape. *R. N. Goss* (San Diego, Calif.).

Marx, G. On the variational principle of dielectrics. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 29–35.

The author discusses some integrals of invariant functions over regions of space-time and the variation of these integrals with respect to the metric tensor. The tensors thus derived are called stress-energy tensors and their properties are discussed. The physical significance of the results obtained is not clear. *A. H. Taub.*

Nicolau, Edmond. Les relations différentielles linéaires entre les équations des potentiels du champ électromagnétique. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **2** (1953), no. 3, 148–155. (Romanian. Russian and French summaries)

En appliquant la méthode des matrices associées aux systèmes d'équations aux dérivées partielles au système d'équations des potentiels électromagnétiques dans le vide, on établit d'abord les relations différentielles linéaires, qui existent entre les équations du champ de Maxwell. Par les mêmes méthodes, on peut démontrer ensuite la possibilité d'intégrer les équations qui relient les potentiels électromagnétiques dans un milieu homogène et isotrope, dans lequel existent un courant de conduction et des charges électriques libres, ainsi que l'intégrabilité des équations attachées au photon par L. de Broglie.

Les rélations différentielles linéaires des potentiels électromagnétiques sont:

$$\frac{\partial E_1}{\partial x}+\frac{\partial E_2}{\partial y}+\frac{\partial E_3}{\partial z}-\mu\varepsilon\frac{\partial E_4}{\partial t}=0,$$

$$\frac{\partial E_5}{\partial x}+\frac{\partial E_6}{\partial y}+\frac{\partial E_7}{\partial z}-\mu\varepsilon\frac{\partial E_8}{\partial t}=0,$$

où E_1-E_8 sont les équations des potentiels et antipotentiels électromagnétiques. *Résumé de l'auteur.*

Nicolau, Edmond. Conservation de l'énergie et de l'impulsion dans les milieux à viscosité électromagnétique. Com. Acad. R. P. Romîne **5** (1955), 1457–1461. (Romanian. Russian and French summaries)

Tihonov, A. N.; and Samarskiĭ, A. A. On the magnetization of a cylinder with winding taking magnetic viscosity into account. Ž. Tehn. Fiz. **25** (1955), 2319–2328. (Russian)

The magnetic induction and the flux of the magnetic induction through a cross-section of the cylinder are

found for the following problem: An infinite conducting cylinder parallel to the z-axis is initially in a constant magnetic field H_0 directed along the z-axis. At the instant $t=0$ the external field changes abruptly in magnitude from H_0 to H_1. Both the elastic and viscous magnetization are taken into account . Another section of the paper deals with the countereffect of the current induced in the winding by the change in the magnetic field on the magnetic field inside the cylinder. The paper is extremely difficult to read for anyone not acquainted with the Russian literature on the subject. For example, the meaning of "viscous magnetization" is not clear. Several Western texts consulted, such as Smythe "Static and dynamic electricity" [2nd ed., McGraw-Hill, New York, 1950] or Durand "Electrostatique et magnétostatique" [Masson, Paris, 1953; MR **16**, 99], discuss, in addition to permanent magnetization, only the magnetization defined by $I_0=\varkappa H$, where $\varkappa$ is the magnetic susceptibility. I_0 is the quantity called elastic magnetization in the present paper. *J. E. Rosenthal* (Passaic, N.J.).

Driver, Rodney D. The undetermined constant in relative-velocity electrodynamics. Discussion by Parry Moon and Domina Eberle Spencer. J. Franklin Inst. **261** (1956), 547–549; discussion, 550–551.

Birkhoff, Garrett; and Diaz, J. B. Non-linear network problems. Quart. Appl. Math. **13** (1956), 431–443.

Starting from first principles, the authors present a clear and systematic treatment of the fundamental problems of the steady flow of current in an electrical (or thermal, or hydraulic) network. Ohm's law for the branches of the network is replaced by a monotone functional relation between current and potential difference. This generalized Ohm's law was introduced by the reviewer in order to preserve the uniqueness properties possessed by linear networks. [Bull. Amer. Math. Soc. **52** (1946), 833–838; MR **8**, 244]. In analogy with potential theory various boundary-value problems are given the self-explanatory names Dirichlet, Neumann and mixed problems. They state that their main result is a general relaxation procedure for the mixed boundary-value problem. One starts with a trial solution for the node potentials which satisfies the boundary conditions. Then point relaxation node by node leads to the solution provided every node occurs infinitely often. This procedure is suited to numerical analysis; it generalizes a result of Diaz and Roberts [Quart. Appl. Math. **9** (1952), 355–360; MR **13**, 388]. *R. J. Duffin* (Pittsburgh, Pa.).

Talbot, A. Some fundamental properties of networks without mutual inductance. Proc. Inst. Elec. Engrs. C. **102** (1955), 168–175.

It is noted that for real values of the complex frequency a network without mutual inductance reduces to a resistance network. Making use of this fact the author presents simplified proofs for the properties of the transfer function of an RC network found by Fialkow and Gerst [Quart. Appl. Math. **10** (1952), 113–127; MR **14**, 116]. *R. J. Duffin* (Pittsburgh, Pa.).

De Claris, Nick. An approximation method with rational functions. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. **287** (1954), 27 pp.

A method is developed for approximating a network function on the imaginary axis by rational functions with prescribed poles. The procedure employs mapping of the right half-plane into the unit circle. An r.m.s. error criterion is employed but the evaluation procedure is by an interpolation technique. *R. J. Duffin.*

Ishida, Tahito. A note on the nonlinear devices of filtering and prediction. Rep. Univ. Electro-Commun. **7** (1955), 49–58. (Japanese summary)

Zadeh [J. Appl. Phys. **24** (1953), 396–404; MR **15**, 376] considered classes of nonlinear filters and derived linear integral equations characterizing filters optimum in the sense of least-squares. When random or gaussian noise is present, following Thomson [Proc. Inst. Elec. Engrs. C. **102** (1955), 46–48; MR **16**, 1036] simplifications in the analytical expressions are to be expected by the use of Hermite polynomials. In this paper the author expresses the probability distribution functions of the signal and the noise as infinite series involving the Hermite polynomials and computes the output of the optimum Zadeh filter. The cases considered in particular include gaussian noise, random noise of infinite bandwidth, and both signal and random noise of limited bandwith. The same produre is also employed to apply the Fourier transform criterion of Van Vleck, Middleton, and North [S. Goldman, Information theory, Prentice-Hall, New York, 1953; MR **16**, 269] in the nonlinear case. Formulas are too involved for inclusion in this review. *S. Kullback.*

Guillemin, E. A. Computational techniques which simplify the correlation between steady-state and transient response of filters and other networks. Proc. Nat. Electronics Confer. **9** (1953), 513–532 (1954).

Greniewski, Henryk; Bochenek, Krystyn; and Marcyński, Romuald. Application of bi-elemental Boolean algebra to electronic circuits. Studia Logica **2** (1955), 7–76. (Polish and Russian summaries)

Exposition des aspects les plus élémentaires de l'application de l'algèbre de Boole aux réseaux d'interrupteurs. *J. Riguet* (Paris).

See also: Shimbel, p. 934; Linn und Pöschl, p. 984; Schmutzer, p. 1015; Popovici, p. 1017.

Quantum Mechanics

Kümmel, Hermann. Irreversibilität und Quantentheorie. Z. Naturf. **11a** (1956), 15–20.

The paper is concerned with the derivation and explanation of classical irreversibility on the basis of quantum mechanics. The author shows what kind of approximations and smoothing-out procedures are necessary in order to obtain classical irreversibility from basic equations which are invariant under time reflection. *J. Werle* (Warsaw).

Case, K. M. Representations of the Duffin-Kemmer algebras. Phys. Rev. (2) **101** (1956), 439–448.

The irreducible representations of the Duffin-Kemmer algebra of dimension n are constructed by means of two sets of spinor matrices belonging to the same dimension n. This is essentially the method of fusion originally developed by L. de Broglie [Théorie générale des particules à spin (méthode de fusion,) Gauthier-Villars, Paris, 1943]. All irreducible representations are obtained anp some properties are derived. The relation between the

Duffin-Kemmer matrices for two successive values of n is established and is used to obtain the subsidiary conditions implied by a Kemmer wave equation. *L. Van Hove.*

Sokolik, G. A. Classification of non-linear equations and relativistically invariant interactions by representations of the Lorentz group, and the fusion theory. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 429–432. (Russian)

The method of the author's earlier paper [Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 817–820; MR **17**, 331] is here applied to the classification of types of non-linear relativistic equations. In particular, the classification is outlined for a single spinor field, and for a Fermi field interacting with a Bose field. The results are shown to be generalizations of those obtained by de Broglie's "method of fusion". *F. J. Dyson* (Princeton, N.J.).

Mickevič, N. V. The scalar field of a stationary nucleon in a nonlinear theory. Ž. Eksper. Teoret. Fiz. **29** (1955), 354–361. (Russian)

The author discusses the classical meson field equation

$$\Box\varphi-(k_0^2+\lambda\varphi^2)\varphi=0$$

with an eye to finding particle-like solutions in which the total field energy is finite. He considers both the case $\lambda>0$, in which the singularities can be regarded as arising from virtual quantum polarization effects and the case $\lambda<0$, which cannot be regarded as the classical limit of a quantum effect. For the case $\lambda<0$, the discussion covers much the same ground as previous work by R. Finkelstein et al. [Phys. Rev. (2) **83** (1951), 326–332; MR **13**, 76] but is mathematically more complete. The main innovation is the application of a method of Chaplygin [A new method for the approximate integration of differential equations, Gostehizdat, Moscow-Leningrad, 1950] to study the behavior of the solutions. Using an approximate form of the particle like solution of lowest energy, the total field energy is then calculated. It turns out that for a meson field varying asymptotically as $(g/r)(\exp -k_0 r)$ it is possible to get a total energy equal to the rest energy of the nucleon confined to a region whose size is the meson compton wave length if $g/e\approx 30$, and the nucleon compton wave length $g/e\approx 85$ (e is the electron charge).

A. S. Wightman (Princeton, N.J.).

Klein, Abraham. Derivation of Low scattering formalism. Phys. Rev. (2) **102** (1956), 913–914.

Plebański, Jerzy. On certain wave-packets. Acta Phys. Polon. **14** (1955), 275–293. (Russian summary)

A detailed examination of simple harmonic oscillator wave-functions is made, generalizing the Kennard wave-packet properties. The results are then extended to the case of a quantized free field, a superposition of such oscillators. The main results presented pertain to the behavior of an arbitrary state-vector under translation and scale-change of the coordinates and momenta, relating the effects of translation to the classical variables and their initial values, and the scale-changes to the mean-square deviations. The phase difference between the original and translated vectors is expressed in terms of corresponding classical solutions. An abbreviated version of this paper has also been published in Phys. Rev. (2) **101** (1956), 1825–1826. *S. Deser.*

Coish, H. R. Infeld factorization and angular momentum. Canad. J. Phys. **34** (1956), 343–349.

The orbital angular momentum operators $L_x \pm iL_y$ associated with problems involving spherical harmonics are closely related to the "ladder" operators of the factorization method. This idea is extended and exploited so that the factorizations for the symmetric top, the free particle on a hypersphere, and the Kepler problem are derived from their angular momentum properties. (The Kepler problem must first be interpreted as a four-dimensional angular-momentum problem.) The electromagnetic pole problem, Weyl's spherical harmonics with spin, and the particle in a space of constant negative curvature can be treated in the same way. It is also shown that this procedure yields, in an elementary way, the momentum-space wave functions for the Kepler problem.

T. E. Hull (Vancouver, B.C.).

Heisenberg, W. Erweiterungen des Hilbert-Raums in der Quantentheorie der Wellenfelder. Z. Physik **144** (1956), 1–8.

A new interpretation is here presented of the "Hilbert-Space II" which was postulated in the author's theory of elementary particles [W. Heisenberg, F. Kortel and H. Mittler, Z. Naturf. **10a** (1955), 425–446; MR **17**, 330; and earlier papers there cited]. It is proposed that Hilbert-Space II is built of "dipoles", each dipole consisting of one state with positive metric and one with negative metric. The two states of a dipole do not have an independent existence, but always combine in such a way as to eliminate divergences from the theory. Arguments of an intuitive kind are presented, showing that the transition matrix elements from a physical state to a dipole state are always zero, so that the physical S-matrix remains unitary. The whole theory is described very briefly and without much mathematical detail. *F. J. Dyson.*

Potier, Robert. Sur une présentation explicitement covariante de la seconde quantification. C. R. Acad. Sci. Paris **242** (1956), 470–473.

A few well-known facts are recalled on the state vectors of relativistic particles, their representation by means of basic vectors and the corresponding representation of state vectors representing systems of several such particles. *L. Van Hove* (Utrecht).

Potier, Robert. Sur la conception tensorielle de la seconde quantification. C. R. Acad. Sci. Paris **242** (1956), 878–881.

In continuation of the paper reviewed above, creation and annihilation operators of the usual type are introduced in Fock space. [See also J. M. Cook, Trans. Amer. Math, Soc. **74** (1953), 222–224; MR **14**, 825.]

L. Van Hove (Utrecht).

Potier, Robert. Sur la théorie générale des champs quantifiés: opérateurs, équation d'évolution, représentation de Schrödinger. C. R. Acad. Sci. Paris **242** (1956), 1961–1964.

Freistadt, Hans. Quantized field theory in the Hamilton-Jacobi formalism. Phys. Rev. (2) **102** (1956), 274–278.

The author says in the introduction that he wants to "modify the Hamiltonian-Jacobi formalism for classical field theory, previously developed by the author [Phys. Rev. (2) **97** (1955), 1158–1161; MR **16**, 778], in a manner analogous to that used by Bohm to obtain, from the Schrödinger equation for a particle, a modified Hamilton-Jacobi equation which in principle describes the motion of the particle in the sense of classical mechanics". This

means that the author writes the state vector ψ of the system in the form $\psi=\exp(\chi-iS/\hbar)$, where χ and S are real functionals of the field operators and of some space-like surface σ (i.e. essentially of a time coordinate). Actually, these two quantities are written as integrals over the surface σ of certain density functionals. Rules are given how to construct these functionals when the Lagrangian of the system is known and how to write down a coupled system of equations between χ and S. The connection with more orthodox formulations of the problem is discussed with the aid of the Klein-Gordon equation for a free field, but no explicit comparison of the solution obtained by the new method with the well-known conventional solution is given. *G. Källén* (Copenhagen).

Bocchieri, P.; e Loinger, A. Su una formulazione hamiltoniana covariante della teoria classica dei campi. Nuovo Cimento (10) **2** (1955), 1058–1062.

A classical analogue of the Yang-Feldman [Phys. Rev. (2) **79** (1950), 972–978; MR **12**, 569] and Tomonaga-Schwinger [Tomonaga, Progr. Theoret. Phys. **1** (1946), 27–42; MR **10**, 226; Schwinger, Phys. Rev. **74** (1948), 1439–1461; MR **10**, 345] theory is worked out. The method consists in giving a Hamiltonian functional, $\mathcal{H}(\sigma, x)$, (depending on a space-like surface, σ, and a space-time point x) as function of field variables, and the Poisson brackets of the field variables. The rate of change of a field variable with respect to changes in σ can then be computed as the Poisson bracket of the field variable with $\mathcal{H}(\sigma, x)$. The example of a complex spin-zero field interacting with the electromagnetic field is worked out in detail. *A. S. Wightman* (Princeton, N.J.).

Grawert, Gerald. Die Lorentz-invariante Zwei-Teilchengleichung und Darstellungen der Lorentz-Gruppe. Z. Physik **143** (1955), 435–450.

The requirement of form-invariance under the inhomogeneous Lorentz group imposes certain general restrictions on integral equations of the Bethe-Salpeter type [Phys. Rev. (2) **84** (1951), 1232–1242; MR **14**, 707]. The author studies these restrictions from the group-theoretical point of view. A functional Hilbert space with an L_2-norm is defined in the (4-dimensional) space of the relative momentum-energy variables of the two particles, and the major part of the paper is concerned with the theory of the reduction of this space by use of the unitary representations of the homogeneous Lorentz group. An attempt is made to show on general grounds that equations of the Bethe-Salpeter type can be expected to have both discrete and continuous parts to their spectra, and that the discrete part will be associated with bound states when the wave function is interpreted in terms of the space-time variables. The difficulties of physical interpretation which are associated with the definition of the Hilbert space and with the relative space-time variables of the particles are noted by the author but not discussed in detail. The paper is of particular interest for its effort to place the unitary representations of the Lorentz group on a basis of practical application. *E. L. Hill.*

Nagy, K. L. On the deduction of the Dirac-Fock-Podolski equations from the quantum theory of fields. Acta Phys. Acad. Sci. Hungar. **6** (1956), 143–147.

Morpurgo, G.; and Touschek, B. F. Remarks on time reversal. Nuovo Cimento (10) **1** (1955), 201–204.

In a previous paper [G. Morpurgo, B. F. Touschek and L. A. Radicati, Nuovo Cimento (9) **12** (1954) 677–698; MR **17**, 438] it was shown that a system of meson and nucleon fields with both scalar and vector couplings is not invariant under time-reversal. In this paper the earlier argument is criticized and a fully rigorous proof of the same result is supplied. *F. J. Dyson.*

Bonč-Bruevič, V. L. Adiabatic approximation in the theory of the Green's function. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 689–692. (Russian)

The Green's functions of a pair of interacting fields (one with Fermi and one with Bose statistics) can be expressed as functional integrals [see e.g., N. N. Bogolyubov, same Dokl. (N.S.) **99** (1954), 225–226; MR **16**, 778]. The author here proposes to evaluate such integrals approximately by making an "adiabatic approximation" which consists in the neglect of functional derivatives of the Green's functions with respect to the Boson field. Thus it is assumed that the state-functional is in some sense a "slowly varying" functional of the Boson field variables. The formal consequences of this assumption are deduced, but no concrete evaluations of the Green's functions are attempted. *F. J. Dyson* (Princeton, N.J.).

Blank, V. Z. Behavior of the Green's function of the electron for small momenta. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 706–709. (Russian)

In quantum electrodynamics, in addition to the ultraviolet catastrophe which leads to infinite corrections to the mass and charge of the electron there is the infra-red catastrophe which arises from the inapplicability of perturbation theory to processes involving large numbers of low-energy photons. In this paper, the Schwinger equation [Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 452–455; MR **13**, 520] for the electron Green's function is solved by a method which is approximate but not dependent on ordinary perturbation theory. The main result is that the renormalized solution is of the form

$$[\mathbf{k}-m]^{-1}[1-m^2/k^2]^{-3\alpha/2\pi}$$

for $k^2\approx m^2$, where m is the renormalized mass of the electron, k is its four-momentum, $\mathbf{k}$ is the scalar product of k with the Dirac matrice and α is the fine structure constant. This formula shows that the Green's function of the coupled field has a worse singularity than that of the free field. *A. S. Wightman* (Princeton, N.J.).

Nambu, Yoichiro. Structure of Green's functions in quantum field theory. II. Phys. Rev. (2) **101** (1956), 459–467.

The author in an earlier paper [Phys. Rev. (2) **100**, (1955), 394–411; MR **17**, 440] found analytic representations of Green's functions formed by taking vacuum expectation values of products of 2, 3 or 4 field operators. Here the analysis is extended to products of any number of field operators. The formulae obtained are of the same general type as before, but are too complicated to be quoted. *F. J. Dyson* (Princeton, N.J.).

Nambu, Yoichiro. Renormalization constants. Phys. Rev. (2) **101** (1956), 1183–1186.

The author presents an argument to the effect that the renormalization constant Z_1 associated with the the vertex function must be zero in symmetric, pseudoscalar meson theory. The proof consists of a manipulation of the spectral representations of many-particle Green's functions given earlier by the author [see the paper

reviewed above] and is therefore essentially based on the correctness of these representations. In the reviewer's opinion, the spectral representations given are not the most general ones (the author has informed the reviewer that he is in general agreement with this statement) and the proof is therefore not quite conclusive. Nevertheless, the reviewer agrees with the conclusion that Z_1 is zero as an entirely independent argument leading to this result and based on the absence of anomalous states with "negative probabilities" in the theory can be constructed.

G. Källén (Copenhagen).

Gor'kov, L. P. On the asymptotic form of the Green's function of an electron. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 65–68. (Russian)

Let the electron Green's function in quantum electrodynamics be represented in the form

$$G(p)=\beta(p^2)[\hat{p}-m]^{-1}.$$

L. D. Landau, A. A. Abrikosov and I. M. Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 1177–1180; MR **16**, 316] calculated the asymptotic form of the Green's functions for large momenta. They, in fact, calculated $\beta(p^2)$ retaining all terms of the form $e^{2n}[\log p^2]^s$ with $s=n$, but neglecting terms with $s<n$. To this accuracy, with a particular choice of gauge, they found the result $\beta(p^2)=1$. In the present paper the calculation is extended so as to include all terms with $s\geqq n-1$. To this accuracy the result is

$$\beta(p^2)=$$
$$1-(e^4/32\pi^2)\log(-p^2/m^2)[1-(e^2/3\pi)\log(-p^2/m^2)]^{-1}.$$

The derivation makes use of the general asymptotic formulae of M. Gell-Mann and F. Low [Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315]. *F. J. Dyson.*

Kastler, Daniel. Sur la représentation usuelle des règles de commutation en théorie quantique des champs. C. R. Acad. Sci. Paris **242** (1956), 1132–1134.

The Fock representation of systems of Bose or Fermi particles is introduced by considering the symmetrized or antisymmetrized direct products of one particle spaces. [See also J. M. Cook, Trans. Amer. Math. Soc. **74** (1953), 222–245; MR **14**, 825.] *L. Van Hove* (Utrecht).

Logunov, A. A. Vertex part in scalar electrodynamics in the region of large momenta. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 223–225. (Russian)

The author calculates the asymptotic form, in the region of high momenta, of the vertex function of scalar quantum electrodynamics. The analogous calculation for spinor electrodynamics was made by V. V. Sudakov in the paper reviewed below. The results are very similar in the two cases. The author's calculation is, however, carried through much more simply and elegantly. He uses the method described in his earlier paper [Z. Eksper. Teoret. Fiz. **29** (1955), 871–874]. *F. J. Dyson.*

Sudakov, V. V. Vertex parts for very high energies in quantum electrodynamics. Z. Eksper. Teoret. Fiz. **30** (1956), 87–95. (Russian)

The asymptotic form of the "vertex" Green's function $\Gamma_s(p,q;l)$ is calculated, assuming that the electron momenta p and q are "almost free" while the photon momentum l is very far from the light-cone. This means that Γ_s describes the elastic scattering of an electron by a potential, with very large transfer of momentum. The calculation is straightforward, using Feynman integrals, and is carried to all orders of perturbation theory. It is found that the contribution from the $2n$th order is proportional to $e^{2n}[\log(l^2/m^2)]^{2n}$. This disagrees with the result of L. D. Landau, A. A. Abrikosov and I. M. Halatnikov [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 1177–1180; MR **16**, 316] who found a result of order $e^{2n}[\log(l^2/m^2)]^n$. The method of the three authors is only valid in the physically less interesting case when all three of the momenta p, q, l are far off the light-cone.

F. J. Dyson (Princeton, N.J.).

Bogolyubov, N. N.; and Širkov, D. V. A model of Lie type in quantum electrodynamics. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 685–688. (Russian)

This paper consists of two parts. Part 1. The argument by which I. Ya. Pomerančuk [same Dokl. (N.S.) **103** (1955), 1005–1008; MR **17**, 565] found an inconsistency in quantum electrodynamics leads without any complicated calculation to the result $Z_3=-\infty$, where Z_3 is the charge-renormalization constant which for a well-behaved theory must lie in the range $0<Z_3<1$. This shows that Pomerančuk's limiting process leads to a theory having precisely the same pathological behavior as the model of T. D. Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR **16**, 317]. Part 2. Pomerančuk's argument is discussed and criticized. The authors conclude that Pomerančuk's limiting process is not the only way to define a quantum electrodynamics with point interaction. Their verdict concerning the inconsistency of quantum electrodynamics is "not proven". *F. J. Dyson* (Princeton, N.J.).

Širkov, D. V. The two-charge renormalization group in pseudo-scalar meson theory. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 972–975. (Russian)

N. N. Bogolyubov and D. V. Širkov [same Dokl. (N.S.) **103** (1955), 203–206; MR **17**, 441] obtained exact differential equations, which are satisfied by the Green's functions of quantum electrodynamics in consequence of the invariance of the theory under transformations of the "renormalization group". The work is here extended to the pseudoscalar meson theory with two independent coupling constants. The two interactions are assumed to be of the form $g\bar{\psi}\gamma_5\psi\phi$ and $\lambda\phi^4$. The results of the analysis are exactly the same, except for different numerical coefficients, as in quantum electrodynamics.

F. J. Dyson (Princeton, N.J.).

Pomerančuk, I. On renormalization of meson charge in pseudoscalar theory with pseudoscalar coupling. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 51–53. (Russian)

In this note, it is argued that the quantum field theory of pseudo-scalar mesons coupled to nucleons by pseudo scalar coupling is inconsistent except when the renormalized coupling constant is zero. To reach this conclusion the author studies the behavior of Green's functions of the nucleon and meson in the theory with double cut-off of Abrikosov and Halatnikov [same Dokl. (N.S.) **103** (1955), 993–996; MR **17**, 565]. Diagrams containing crossed meson lines and meson-meson scattering are neglected in the calculations. An expression of the type

$$g^2=g_1^2[1+\pi^{-1}g_1^2(L_p+\tfrac{1}{4}L_k]^{-1}$$

is obtained connecting the unrenormalized coupling constant, g_0, with the renormalized coupling constant g. Here L_p and L_k are the nucleon and meson cut-offs respectively. When L_p and L_k approach infinity g^2

approaches zero. The argument runs parallel to previous work by the author on electrodynamics [ibid. **103** (1955), 1005–1008; MR **17**, 765]. *A. S. Wightman.*

Pomerančuk, I. On the vanishing of the renormalized meson charge in pseudoscalar theory with pseudoscalar coupling. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 461–464. (Russian)

This note refines the argument of the paper reviewed above. The previously neglected contributions to the Green's functions arising from crossed diagrams and meson-meson scattering are discussed and after a rather involved argument the same conclusion as before is reached. The author expresses the opinion that the above work shows the necessity of introducing a new length r_0 into the theory. He points out that the value of r_0 is restricted in his theory by the requirement that the renormalized coupling constant agree with that determined from experiments on photo-meson production. The restriction seems inconsistent with other restrictions deduced by Landau from cosmic ray evidence.

A. S. Wightman (Princeton, N.J.).

Brodskiĭ, A. M. On renormalization in mesodynamics. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 939–942. (Russian)

Schwinger showed [Phys. Rev. (2) **82** (1951), 664–679; MR **12**, 889] how the interaction of the electron-positron vacuum with an external unquantized electromagnetic field may be compactly described using Lagrangian techniques. In the present note Schwinger's method is extended to the case of a general set of fields by the use of an auxiliary external gravitational field. The resulting expression for the contribution to the action is then written out in the case of pseudoscalar mesons interacting with nucleons through symmetrical pseudoscalar and pseudo vector coupling. An expansion of the action in powers of the meson field strength then gives expressions for the mass, coupling constant and field strength renormalization constants in terms of integrals over a proper time parameter. As in the electromagnetic case, the integrals diverge because the contributions from small proper times are too large. *A. S. Wightman.*

Heber, Gerhard. Zur Theorie der Elementarteilchen. II. Ann. Physik (6) **17** (1956), 102–108.

This paper is primarily devoted to a discussion of the author's unquantized field theory of extended particles [Ann. Phys. (6) **16** (1955), 43–51; MR **17**, 566]. According to that formalism a nucleon consists of a bare nucleon (represented by a non-relativistic wave packet of finite internal kinetic energy) surrounded by a relativistic meson cloud with finite binding energy. Though unobservable, the bare nucleon seems to play an essential role in the theory. The fact that one is here not dealing with point particles prohibits the usual canonical field quantization and the author suggests that a modification of the usual quantization procedure is essential for a finite theory. *F. Rohrlich* (Iowa City, Ia.).

Heber, Gerhard. Zur Theorie der Elementarteilchen. III. Quantenfeldtheorie für ausgedehnte Nukleonen. Z. Physik **144** (1956), 39–55.

The author's theory of extended elementary particles [Ann. Physik (6) **16** (1955), 43–51; MR **17**, 566; see also the paper reviewed above] is quantized. The quantization differs from the canonical quantization in two essential points: (1) The commutation relations for equal times do not contain a Dirac δ-function, but a regular function. (2) Certain degrees of freedom of the system are suppressed. The starting point is the non-relativistic Hamiltonian,

$$H=-\frac{\hbar^2}{2M}\int\psi^*\nabla^2\psi d^3\tau-\frac{g^2}{2}\iint\psi^*(\tau)\psi(\tau)d^3\tau G(\tau-\tau')d^3\tau'\psi^*(\tau')\psi(\tau')$$

with $G(\tau)=e^{-\mu|\tau|}/4\pi|\tau|$ and the quantization

$$[\psi(\tau),\psi^*(\tau')]=\zeta(\tau,\tau'),\ [\psi(\tau),\psi(\tau')]=[\psi^*(\tau),\psi^*(\tau')]=0,$$

where $\zeta(\tau,\tau')$ is finite at $\tau=\tau'$. It is now evident that the above two modifications of the canonical quantization procedure are in a sense closely related: Let $\{\varphi_n(\tau)\}$ be a complete orthonormal set of functions; then $\delta(\tau-\tau')=\sum_n\varphi_n^*(\tau)\varphi_n(\tau')$. If we choose $\zeta(\tau,\tau')=\sum_{n=0}^{n_0}\varphi_n^*(\tau)\varphi_n(\tau')$, then the expansion coefficients in $\psi(\tau)=\sum_n c_n\varphi_n(\tau)$ are quantized for $n\leqq n_0$, but unquantized for the "degrees of freedom" $n>n_0$. One can now suppress the unquantized c_n by assuming $c_n=0$ for $n>n_0$. This procedure permits a cut-off at large momenta.

A special case is considered in detail where to each function $\varphi_n(r)$ $(n=0, 1, \cdots, n_0)$ there is associated a point $\mathfrak{R}_n$ which is the position of its maximum and in whose vicinity only $\varphi_n(\tau)$ is non-vanishing. Two interpretations then appear possible: Either the system contains a finite number of particles whose finite extension is characterized by $\varphi_n(\tau-\mathfrak{R}_n)$, or the system contains point particles which cannot be sharply localized. The second alternative derives from the impossibility in this theory to create or destroy point particles (i.e. completely localized states). The connections with other lattice-space theories are pointed out. Schrödinger equations for one and two-particle systems are derived. They correspond to linear integral equations *F. Rohrlich* (Iowa City, Ia.).

Meyer, Klaus. Zur Durchführung eines Variationsverfahrens in einer skalaren Feldtheorie. Ann. Physik (6) **17** (1956), 109–114.

Heber's model of a nucleon as a scalar relativistic meson cloud coupled to a scalar non-relativistic bare nucleon [Ann. Physik (6) **16** (1955), 43–51; MR **17**, 566] leads to the following equation for the total energy ΔW in terms of the nucleon momentum distribution $b(\mathfrak{p})$:

$$\Delta W=\frac{\hbar^2}{2M}\int\mathfrak{p}^2b^*(\mathfrak{p})b(\mathfrak{p})d\tau_{\mathfrak{p}}-\frac{g^2}{2(2\pi)^3}\int F^*(\mathfrak{k})F(\mathfrak{k})\frac{d\tau_{\mathfrak{k}}}{\omega_{\mathfrak{k}}^2}$$

with $F(\mathfrak{k})=\int b^*(\mathfrak{p})b(\mathfrak{p}-\mathfrak{k})d\tau_{\mathfrak{p}}$ and $\omega^2_{\mathfrak{k}}=\mu^2+\mathfrak{k}^2$. The symbols μ, M, and g denote the meson mass, the bare nucleon mass, and the coupling constant. The problem is to find a $b(\mathfrak{p})$ which minimizes ΔW and is normalized to

$$\int b^*(\mathfrak{p})b(\mathfrak{p})d\tau_{\mathfrak{p}}=1.$$

It is shown that such a $b(\mathfrak{p})$ exists and that it leads to a finite minimum of ΔW. It is also proven that it satisfies $\lim_{|\mathfrak{p}|\to\infty}|b(\mathfrak{p})|\,|\mathfrak{p}|^n=0$ for $n\geqq 0$. From examples it appears that $(\Delta W)_{\min}$ will be negative only when $g^2/\hbar c$ is larger than a certain minimum value. *F. Rohrlich.*

Ahiezer, A.; Aleksin, V.; and Volkov, D. On some effects resulting from the interaction of an electromagnetic field with a vacuum of scalar charged particles. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 830–833. (Russian)

Following the methods of F. Rohrlich and R. Gluck-

stern [Phys. Rev. (2) **86** (1952), 1–9], the scattering of light by light is calculated for the electrodynamics of spin-zero particles. The forward coherent scattering of light by a Coulomb field is also calculated. The results differ only by numerical coefficients from the results obtained by Rohrlich and Gluckstern for particles with spin $\frac{1}{2}$. *F. J. Dyson* (Princeton, N.J.).

Moshinsky, Marcos. Diffraction in time and Dirac's equation. Rev. Mexicana Fisica **1** (1952), 151–164. (Spanish English summary)

The problem previously treated by the author for spinless particles [Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina, Rio de Janeiro and São Paulo, 1952, Rio de Janeiro, 1954, pp. 201–215; MR **16**, 655] is discussed for Dirac particles. The Dirac spinors which describe the phenomenon which arises when a shutter is opened for a beam of plane waves are expressed in terms of Lommel functions. Their asymptotic behaviour on the light cone and as t approaches infinity and their behaviour under Lorentz transformations are discussed.

A. J. Coleman (Toronto, Ont.).

Halbwachs, Francis. Modèle de fluide relativiste décrivant les ondes de Kemmer pour le corpuscule de spin 0. C. R. Acad. Sci. Paris **242** (1956), 2106–2109.

Amat, Gilbert; Goldsmith, Mark; and Nielsen, Harald H. Sur le commutateur de deux polynomes homogènes. Application au calcul de l'énergie de vibration d'une molécule polyatomique. J. Phys. Radium (8) **16** (1955), 854–860.

The computation of the commutators of two polynomials in certain problems in physics leads to the evaluation of a great number of simple commutators. The authors propose an alternative method based on recurrence relations. This is carried out for monomials and for general homogeneous polynomials and then applied to the problem mentioned in the title.

O. Taussky-Todd (Washington, D.C.).

Löwdin, Per-Olov. Quantum theory of cohesive properties of solids. Advances in Physics **5** (1956), 1–172.

This paper is a comprehensive review of the theory of the quantum states of electrons in perfect crystals. It covers the states of a single electron in the periodic field of a crystal, the fitting together of these into Hartree-Fock self-consistent field solutions and the evaluation of their energies, and the full many-electron problem. In all these areas the author, by design, emphasizes those approaches which start from the properties of free atoms, e.g., those based on molecular orbitals constructed by linear combination of atomic orbitals. The bibliography is very complete, however, nearly all papers based on other approaches being at least mentioned. Among the topics given particularly extensive treatment are: The use of one-particle, two-particle, etc., density matrices in general many-body problems; the treatment of electrostatic interactions; procedures for constructing an orthonormal set from a given set of basis functions; the explicit construction of orthonormal Bloch waves from atomic functions; and the evaluation of the integrals which occur in the expression for the energy of a determinant of such states. Specific applications to the calculation of binding energy, lattice constant, and elastic constants are discussed, both for metals and for ionic crystals. It is pointed out that for the latter a representation of the electronic energy in terms of interactions between atoms requires many-body, rather than merely two-body potentials. *C. Herring* (Murray Hill, N.J.).

See also: v. Weizsäcker, p. 932; David, p. 989; Sharp, Gore and Paul, p. 1013; Shaw, p. 1016; Hanus, p. 1018. Weber, p. 1036.

Thermodynamics, Statistical Mechanics

Biot, M. A. Thermoelasticity and irreversible thermodynamics. J. Appl. Phys. **27** (1956), 240–253.

§ 2 and § 3 develop linearized thermo-elasticity along standard lines. § 4 establishes an analogy with the author's theory of filtration [same J. **12** (1941), 155–164]. § 5 gives the equations for the displacement and the entropy; as is obvious from the first principles and the linearization employed by the author, the entropy satisfies a diffusion equation. The author writes down a formal solution of the Boussinesq-Papkovitch type. § 6 shows that the differential equations are equivalent to a principle of minimum entropy production, being a special case of a variational principle formulated by the author [Phys. Rev. (2) **97** (1955), 1463–1469; MR **16**, 1189]. §§ 7–8 calculate the free energy and give applications of the author's previous operational treatment. §§ 9–10 generalize the formulation to the anisotropic and dynamical cases. The rest of the paper gives some simple applications. *C. Truesdell.*

Schäffer, Juan J. Some problems relative to the chemical equilibrium of a gaseous phase. Bol. Fac. Ingen. Agrimens. Montevideo **5** (1954), 95–118. (Spanish)

Sufficient conditions are established for: (1) a gaseous phase to have one and only one state of chemical equilibrium (and the nature of this state is specified); (2) a gaseous phase to have at least one state of stable equilibrium (and the nature of this stability is specified).

C. C. Torrance (Monterey).

★ **Einstein, Albert. Investigations on the theory of the Brownian movement.** Edited with notes by R. Fürth. Translated by A. D. Cowper. Dover Publications, Inc., New York, 1956. vi+122 pp. $1.25.

Photo-offset reprint of the edition of 1926 [Methuen, London].

Kümmel, H. Zur quantentheoretischen Begründung der klassischen Physik. II. Statistische Mechanik und Thermodynamik. Nuovo Cimento (10) **2** (1955), 877–897.

After having derived in part I [Nuovo Cimento (10) **1** (1955), 1057–1078; MR **17**, 222] the classical hydrodynamical equations of motion on the basis of quantum mechanics, the author is now concerned with the derivation of classical statistical and thermodynamical relations. The main value of the paper consists in clear statements of assumptions and restrictions imposed on the quantum systems to be treated classically. The paper may be, therefore, a starting point for a discussion concerning the range of validity of classical theories and deviations of various macroscopic systems from classical behaviour.

J. Werle (Warsaw).

Pathria, R. K. A note on the (relativistic) statistical mechanics of an assembly in mass-motion. Proc. Nat. Inst. Sci. India. Part A. **21** (1955), 331–337 (1956).

Onicescu, O. L'évolution de la répartition des grandeurs additives de la mécanique statistique. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. **4** (1955), no. 6–7, 31–34. (Romanian. Russian and French summaries)

The author studies the distribution of an additive quantity for a mechanical system with a large number of interacting components. The interaction is introduced by means of a function specifying the number of interactions in time t. *J. L. Doob* (Urbana, Ill.).

Weber, J. Fluctuation dissipation theorem. Phys. Rev. (2) **101** (1956), 1620–1626.

Nyquist's relation between voltage fluctuations and the impedance of an electrical circuit has been extended by Callen and Welton [Phys. Rev. (2) **83** (1951), 34–40; MR **13**, 447] to more general quantum-mechanical systems, by means of first-order perturbation theory. It is here remarked that the result must be exact for systems which are known a priori to have power dissipation quadratic in the driving forces. Various forms of this "fluctuation-dissipation theorem" are given explicitly. It is then applied to establish a relation between the vacuum fluctuations of the electromagnetic field and the emission of radiation by a given current distribution. In this way the vacuum expectation values and the commutators of the four components of the vector potential can be found. Finally, the fluctuations in electrical circuits are shown to lead to divergent integrals; an appropriate cutoff procedure is discussed.

N. G. van Kampen (Utrecht).

Jancel, Raymond. Sur le théorème H en mécanique quantique. C. R. Acad. Sci. Paris **242** (1956), 1268–1270.

The quantum-mechanical observables are made macroscopic, which entails a coarse-graining of the density matrix; subsequently the usual proof of the decrease of H is applied.

N. G. van Kampen (Utrecht).

See also: Sato, p. 977; v. Krzywoblocki, p. 1022; Bordoni, p. 1025.

BIBLIOGRAPHICAL NOTES

Soviet Physics. JETP. A Translation of the Journal of Experimental and Theoretical Physics of the Academy of Sciences of the USSR. Published by the American Institute of Physics, 57 East 55 Street, New York 22, New York.

"Soviet Physics is a bi-monthly journal. ... All issues of the Soviet journal after January 1, 1955 will be translated. ... One volume is published annually. Volume 1 (1955) will contain three issues. Subsequent volumes, beginning with January 1956, will contain six issues."

About one quarter of the articles are of a sufficiently definite mathematical character to be reviewed in Mathematical Reviews. The reviews for MR will usually be based on the original Russian article but in some cases it will be possible to take into account the English translation also.

★**Miles, John W. Unsteady supersonic flow.** Air Research and Development Command, Baltimore, Md., 1955. xii+569 pp.

The Director of Research Information, Headquarters of Air Office of Scientific Research, wishes to state (cf. the review in MR **17**, 426) that an adequate supply of this publication is available at nominal cost from the Office of Technical Services, U. S. Department of Commerce, Washington, D.C., under their number PB 111993.

Quality Control and Applied Statistics Abstracts, Volume I, Issue No. 1. Edited by Titchen, R. S.; Rosenthal, A. J., Bollerman, B.; Nistico, F. Interscience Publishers, Inc., New York-London. 1956. 80 pp. Subscription Price: $60.00 per year.

The Service abstracts all new contributions, devoting one or two pages, sometimes more, to each abstract ... so that it should usually be unnecessary to consult the original paper. This first issue, starting June 1956, contains 37 abstracts, and there are to be twelve issues, totalling approximately 1,000 pages annually. A feature of the service is that the abstracts are supplied as loose leaves, which the reader can file in any way he pleases. (From a letter from the publishers).

The list of journals reviewed contains about 350 titles including most of the standard journals in pure and applied mathematics.